Published by Institute of Physics Publishing for SISSA

Received: *January 2, 2006*
Accepted: *May 17, 2006*
Published: *May 26, 2006*

Non-commutative (D)-instantons*

Marco Billó, Marialuisa Frau, Stefano Sciuto and Giuseppe Vallone

Dipartimento di Fisica Teorica, Università di Torino
and Istituto Nazionale di Fisica Nucleare - sezione di Torino
via P. Giuria 1, I-10125 Torino, Italy
E-mail: billo@to.infn.it, frau@to.infn.it, sciuto@to.infn.it, vallone@to.infn.it

Alberto Lerda

Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale
I-15100 Alessandria, Italy, and
Istituto Nazionale di Fisica Nucleare - sezione di Torino
via P. Giuria 1, I-10125 Torino, Italy
E-mail: lerda@to.infn.it

Abstract: We study systems of D3 and D(−1) branes in a NS-NS magnetic background and show that, when the brane configuration is stable, the physical degrees of freedom of the open strings with at least one end-point on the D-instantons describe the ADHM moduli of instantons for non-commutative gauge theories. We also prove that disk diagrams with mixed boundary conditions are the sources for the classical profile of the non-commutative gauge instantons in the singular gauge. We finally compare the string theory description in a large distance expansion with the non-commutative ADHM construction in the singular gauge and find complete agreement at perturbative level in the non-commutativity parameter.

Keywords: D-branes, Brane Dynamics in Gauge Theories, Solitons Monopoles and Instantons, Non-Commutative Geometry.

*Work partially supported by the European Community's Human Potential Programme under contract MRTN-CT-2004-005104 "Constituents, Fundamental Forces and Symmetries of the Universe", and by the Italian M.I.U.R under contract PRIN-2003023852 "Physics of fundamental interactions: gauge theories, gravity and strings".

JHEP05(2006)069

Contents

1. Introduction **1**

2. Gauge instantons from D3/D(–1) systems **3**

3. D3/D(–1) systems in presence of a B-field **7**
- 3.1 The 3/3 strings 8
- 3.2 The (–1)/(–1) strings 11
- 3.3 The (–1)/3 and 3/(–1) strings 11
  - 3.3.1 Spectrum 14
  - 3.3.2 Moduli spectrum in (anti-)self-dual background 15
  - 3.3.3 The issue of stability 18

4. Non-commutative gauge instantons from open strings **20**
- 4.1 The ADHM constraints 20
- 4.2 The instanton profile in a self-dual background 21

5. The ADHM construction for non-commutative instantons in the singular gauge **25**

6. Conclusions **28**

A. Notation and conventions **30**

1. Introduction

Quantum field theories on non-commutative spaces display a very rich spectrum of unusual properties and for this reason they have attracted a wide interest in the last few years. For instance, they contain a minimal distance scale $|\theta|$, provided by the non-commutativity parameter $\theta^{\mu\nu} \sim [x^\mu, x^\nu]$, which naturally cuts off the theory in the UV, though often at the price of peculiar UV/IR mixing effects.

Initially, a lot of work was devoted to the analysis of non-commutative theories in a field-theoretic framework, but an even greater attention was sparkled by the realization that non-commutativity arises most naturally in a string theory set-up. The stringy connection was originally pointed out [1] in the context of (M)atrix theory compactification, but it was subsequently[1] established in a more direct way [3]–[13] by considering open strings in a magnetic field or in a closed string background with a non-trivial flux for the $B_{\mu\nu}$ field of the NS-NS sector.[2]

A very interesting aspect of the non-commutative deformations of gauge theories is the study of their effects on instantons. This is the main subject of this paper. Realizing a four-dimensional U(N) gauge theory by a stack of N coincident D3 branes, instantons with charge k can be obtained by adding k D(-1) branes, also known as D-instantons [17, 18, 8]. This brane realization not only reproduces and physically explains the ADHM [19] construction (see, for instance, ref. [20] and references therein); it also accounts for the profile of the classical solution and for the instanton calculus of correlators from a string theory point of view, as shown in ref. [21], building also on techniques already introduced in ref. [22]. In a trivial background the moduli space of these stringy instantons coincides with that of classical ADHM gauge instantons [8], but when we introduce a non-commutative deformation $\theta_{\mu\nu}$ by turning on a $B_{\mu\nu}$ field, it is deformed and no longer coincides with the classical one unless the background is self-dual (or anti-self-dual in the case of anti-instantons) [8]. In particular, the ADHM constraints are modified in such a way that the small-instanton singularity [23], which corresponds to the possibility of detaching the D-instantons from the world-volume of the D3 branes, is removed. This basically happens because the D3/D(-1) system no longer satisfy a no-force condition and is no longer stable when the background field is not (anti-)self-dual, in perfect agreement with the features of the (anti-)instanton solutions on non-commutative $\mathbb{R}^4$ obtained by extending the usual ADHM construction to the non-commutative gauge theories [24]-[33].

In this paper, we intend to pursue the line of thought of ref. [21], already successfully applied in the case of non anti-commutative deformations [34], and explicitly derive the measure on moduli space and the instanton profile from open string disk amplitudes in presence of the NS-NS B background. In doing so, we retrieve the expected behaviour, but along the way we encounter some crucial subtleties that render the entire construction non-trivial.

After presenting in section 2 a brief review of the stringy ADHM construction, in section 3 we analyze the quantization of the open strings of the D3/D(-1) system in a B background. As is well known, this quantization can be carried out exactly in the RNS formalism[3] for all kinds of open strings (namely those stretching between two D3's, or between two D(-1)'s, or the mixed ones). However, a careful analysis reveals that there exist different possibilities of imposing boundary conditions on the world-sheet fermions that are compatible with the B background. One therefore obtains a number of different open string sectors that is larger than naively expected. Just like in the commutative case, also in the presence of B the physical excitations of open strings with at least one end-point on the D(-1) branes are interpreted as the instanton ADHM moduli; however in the non-commutative case two points should be stressed: first, this identification is

[1]Much ground-work having already been performed, quite earlier, in ref. [2].

[2]Constant fluxes of RR fields lead instead to non anti-commutative theories [14]–[16].

[3]This is in contrast with the non anti-commutative case where the RR background can only be inserted perturbatively, even if, in the end, this turns out to be sufficient, see for instance [34, 35].

possible only for a self-dual background (or for an anti-self-dual background in the case of anti-instantons), i.e. when the D3/D(-1) system is stable; second, the precise relationship between open string states and ADHM moduli is highly non-trivial and actually it requires a suitable combination of the various open string sectors associated with the different types of fermionic boundary conditions mentioned above.

In section 4, we consider a self-dual background and, using the string construction of section 3, prove that, as expected, the moduli space for non-commutative instantons is not modified with respect to the commutative one. Moreover, we show that the leading term in the large distance expansion of the classical solution is generated by the gauge boson emission amplitude from mixed disks, in perfect analogy with the ordinary gauge instantons [21]. The results of these string calculations are then compared with the non-commutative field theory expectations in section 5. Since the classical instanton profile obtained from the mixed disk emission diagram is in the so-called "singular gauge", in order to compare it with the non-commutative ADHM construction we have to describe the latter also in the singular gauge, rather than in the regular one as is usually done in the literature. This is not a trivial task (see, however, ref. [32] for a discussion on this point), but in our context it is enough to write down an instanton profile which solves the ADHM constraints up to some exponentially suppressed contributions, and allows a meaningful and successful comparison with the string results of section 4.

We conclude by considering the D3/D(-1) system in a generic (i.e. non (anti-) self-dual) B background, whose effects can only be treated in a perturbative way. For the 3/3 strings, this approach was in fact exploited already in [5] to show the emerging of a non-commutative gauge theory from string theory amplitudes. For the $(-1)/3$ and the $(-1)/(-1)$ strings things are actually simpler and the computation of mixed open/closed string diagrams with a single B-insertion is sufficient to exhibit the expected deformation of classical instanton moduli space and of the ADHM constraints. Finally, in the appendix we list our notations and conventions.

2. Gauge instantons from D3/D(–1) systems

The ADHM construction [19] of supersymmetric gauge instantons and their moduli space can be derived in full detail from string theory by considering systems of D3 branes and D-instantons (for a review see, for instance, [20]) [4]. In this approach, the auxiliary variables of the ADHM construction correspond to the degrees of freedom associated to open strings with at least one end-point attached to the D-instantons, and the measure on the moduli space as well as the instanton profile can be obtained directly from disk amplitudes [21]. In order to be self-contained, we briefly review this derivation.

We consider type IIB superstrings in the Euclidean space $\mathbb{R}^{10}$ (whose coordinates we label by the indices $M, N = 1, \ldots, 10$) and place N D3-branes along the first four directions (labeled by the indices $\mu, \nu = 1, \ldots, 4$). The six transverse directions are labeled instead by the indices $m, n = 5, \ldots, 10$. Under this SO(10) $\to$ SO(4) $\times$ SO(6) decomposition, the

[4] For anti-instantons one should consider instead anti-D(-1) branes.

string coordinates X^M and ψ^M split as

$$X^M \;\to\; (X^\mu\,,\,X^m) \quad \text{and} \quad \psi^M \;\to\; (\psi^\mu\,,\,\psi^m)\,, \tag{2.1}$$

while the anti-chiral spin fields $S^{\dot{\mathcal{A}}}$ ($\dot{\mathcal{A}} = 1, \ldots, 16$) of the RNS formalism become products of four- and six-dimensional spin fields according to

$$S^{\dot{\mathcal{A}}} \;\to\; (S_\alpha S_A\,,\,S^{\dot\alpha} S^A) \tag{2.2}$$

where the index α (or $\dot\alpha$) denotes positive (or negative) chirality in four dimensions, and the upper (or lower) index A indicates the fundamental (or anti-fundamental) representation of SU(4) $\sim$ SO(6).

The massless sector of the strings with both ends on D3-branes (3/3 strings) comprises the gauge field A_μ, six scalars φ_m and the gauginos $\Lambda^{\alpha A}$ and $\bar\Lambda_{\dot\alpha A}$, which altogether form the $\mathcal{N} = 4$ vector multiplet. Their vertex operators are

$$V_A(p) = \frac{A_\mu(p)}{\sqrt{2}}\,\psi^\mu\,\mathrm{e}^{-\phi}\,\mathrm{e}^{\mathrm{i}\,p\cdot X}\,, \qquad V_\varphi(p) = \frac{\varphi_m(p)}{\sqrt{2}}\,\psi^m\,\mathrm{e}^{-\phi}\,\mathrm{e}^{\mathrm{i}\,p\cdot X} \tag{2.3}$$

in the (-1) superghost picture of the NS sector, and

$$V_\Lambda(p) = \Lambda^{\alpha A}(p)\,S_\alpha S_A\,\mathrm{e}^{-\frac{1}{2}\phi}\,\mathrm{e}^{\mathrm{i}\,p\cdot X}\,, \qquad V_{\overline{\Lambda}}(p) = \overline{\Lambda}_{\dot\alpha A}(p)\,S^{\dot\alpha} S^A\,\mathrm{e}^{-\frac{1}{2}\phi}\,\mathrm{e}^{\mathrm{i}\,p\cdot X} \tag{2.4}$$

in the $(-1/2)$ picture of the R sector. Here ϕ is the boson of the superghost fermionization formulas, p is the longitudinal incoming momentum and the convention $2\pi\alpha' = 1$ has been taken. The vertices (2.3) and (2.4) describe fields in the adjoint representation of U(N), and their scattering amplitudes give rise to the usual $\mathcal{N} = 4$ SYM theory in the field theory limit $\alpha' \to 0$.

As is well-known, the D3-branes break half of the bulk supersymmetries in target space due to the identification between left- and right-moving spin fields enforced at the boundary, i.e.

$$S_\alpha(z)\,S_A(z) = -\,\widetilde{S}_\alpha(\overline{z})\,\widetilde{S}_A(\overline{z})\Big|_{z=\overline{z}}\,, \qquad S^{\dot\alpha}(z)\,S^A(z) = \widetilde{S}^{\dot\alpha}(\overline{z})\,\widetilde{S}^A(\overline{z})\Big|_{z=\overline{z}}\,. \tag{2.5}$$

Let us now add the D(-1) branes. They correspond to imposing Dirichlet boundary conditions on all string coordinates X^M and ψ^M, and enforcing the following identification on the spin fields [21]

$$S_\alpha(z)\,S_A(z) = \widetilde{S}_\alpha(\overline{z})\,\widetilde{S}_A(\overline{z})\Big|_{z=\overline{z}}\,, \qquad S^{\dot\alpha}(z)\,S^A(z) = \widetilde{S}^{\dot\alpha}(\overline{z})\,\widetilde{S}^A(\overline{z})\Big|_{z=\overline{z}}\,. \tag{2.6}$$

By comparison with (2.5), it is clear that the conditions (2.6) break a further half of the bulk supersymmetries, so that only eight supercharges (those with spinor indices of the type $(\dot\alpha A)$) are preserved on both branes.

The open strings with both ends on the D-instantons ($(-1)/(-1)$ strings) do not carry any momentum since there are no longitudinal Neumann directions. Thus, these strings describe *moduli* rather than dynamical fields. In the NS sector there are ten bosonic moduli corresponding to the physical vertices

$$V_a = g_0\, a'_\mu\, \psi^\mu\, \mathrm{e}^{-\phi}\,, \qquad V_\chi = \frac{\chi_m}{\sqrt{2}}\, \psi^m\, \mathrm{e}^{-\phi}\,, \tag{2.7}$$

while in the R sector there are sixteen fermionic moduli whose vertices are

$$V_{M'} = \frac{g_0}{\sqrt{2}}\, M'^{\alpha A}\, S_\alpha S_A\, \mathrm{e}^{-\frac{1}{2}\phi}\,, \qquad V_\lambda = \lambda_{\dot\alpha A}\, S^{\dot\alpha} S^A\, \mathrm{e}^{-\frac{1}{2}\phi}\,. \tag{2.8}$$

In writing the polarizations of these vertices we have adopted the traditional notation; in particular we have distinguished the bosonic moduli into four a'_μ (corresponding to the longitudinal directions of the D3 branes) and six χ_m (corresponding to the transverse directions to the D3's). Furthermore, g_0 is the dimensionful coupling for the effective theory on the D-instantons, which is related to the Yang-Mills coupling on the D3 branes by

$$g_0 = \frac{g_{\rm YM}}{4\pi^2\alpha'}\,. \tag{2.9}$$

Clearly, if $g_{\rm YM}$ is kept fixed when $\alpha' \to 0$ (as is appropriate to retrieve the gauge theory on the D3-branes), then g_0 blows up. Thus, as discussed in [21], suitable factors of g_0, like the ones appearing in (2.7) and (2.8), must be present in the vertex operators to retain non-trivial interactions when $\alpha' \to 0$. As a consequence the moduli acquire non-trivial scaling dimensions which turn out to be the right ones for their interpretation as parameters of an instanton solution [20, 21]. For instance, the a'_μ's in (2.7) have dimensions of (length) and are related to the positions of the (multi)-centers of the instanton. Finally, we recall that since there are k D-instantons, all the above moduli carry Chan-Paton factors of the adjoint representation of $\mathrm{U}(k)$.

Let us now consider the open strings that are stretched between a D3 and a D(-1) brane, i.e. the $3/(-1)$ or $(-1)/3$ strings. They are characterized by the fact that the four longitudinal directions along the D3 branes have mixed Neumann-Dirichlet boundary conditions, while the remaining six transverse directions have Dirichlet-Dirichlet boundary conditions. As for the $(-1)/(-1)$ strings, also in this case there is no momentum, and the string excitations describe again moduli rather than dynamical fields. In the NS sector the physical vertex operators are

$$V_w = \frac{g_0}{\sqrt{2}}\, w_{\dot\alpha}\, \Delta\, S^{\dot\alpha}\, \mathrm{e}^{-\phi}\,, \qquad V_{\overline{w}} = \frac{g_0}{\sqrt{2}}\, \overline{w}_{\dot\alpha}\, \overline{\Delta}\, S^{\dot\alpha}\, \mathrm{e}^{-\phi}\,, \tag{2.10}$$

where Δ and $\overline{\Delta}$ are the bosonic twist and anti-twist operators with conformal weight $1/4$ which change the boundary conditions of the X^μ coordinates from Neumann to Dirichlet and vice-versa by introducing a cut in the world-sheet [36]. The moduli $w_{\dot\alpha}$ and $\overline{w}_{\dot\alpha}$, whose $\mathrm{SO}(4)$ chirality is fixed by the GSO projection, carry Chan-Paton factors, respectively, in the bi-fundamental representations $\mathbf{N}\times\mathbf{k}$ and $\overline{\mathbf{N}}\times\overline{\mathbf{k}}$ of the gauge groups. Thus, one should

JHEP05(2006)069

	$SO(4) \simeq SU(2)_+ \times SU(2)_-$	$SO(6) \simeq SU(4)$	$U(N)$	$U(k)$	dimensions
a'_μ	$(\mathbf{2},\mathbf{2})$	$\mathbf{1}$	$\mathbf{1}$	**adj**	$(\text{length})^1$
χ_m	$(\mathbf{1},\mathbf{1})$	$\mathbf{6}$	$\mathbf{1}$	**adj**	$(\text{length})^{-1}$
$M'^{\alpha A}$	$(\mathbf{2},\mathbf{1})$	$\mathbf{4}$	$\mathbf{1}$	**adj**	$(\text{length})^{1/2}$
$\lambda_{\dot\alpha A}$	$(\mathbf{1},\mathbf{2})$	$\overline{\mathbf{4}}$	$\mathbf{1}$	**adj**	$(\text{length})^{-3/2}$
$w_{\dot\alpha}$	$(\mathbf{1},\mathbf{2})$	$\mathbf{1}$	$\mathbf{N}$	$\mathbf{k}$	$(\text{length})^1$
$\overline{w}_{\dot\alpha}$	$(\mathbf{1},\mathbf{2})$	$\mathbf{1}$	$\overline{\mathbf{N}}$	$\overline{\mathbf{k}}$	$(\text{length})^1$
μ^A	$(\mathbf{1},\mathbf{1})$	$\mathbf{4}$	$\mathbf{N}$	$\mathbf{k}$	$(\text{length})^{1/2}$
$\overline{\mu}^A$	$(\mathbf{1},\mathbf{1})$	$\mathbf{4}$	$\overline{\mathbf{N}}$	$\overline{\mathbf{k}}$	$(\text{length})^{1/2}$
D^-_c	$(\mathbf{1},\mathbf{3})$	$\mathbf{1}$	$\mathbf{1}$	**adj**	$(\text{length})^{-2}$

Table 1: Transformation properties and scaling dimensions of the ADHM moduli.

write more explicitly $w_{\dot\alpha}{}^{iu}$ and $\overline{w}_{\dot\alpha ui}$, where $u = 1,\ldots,N$ and $i = 1,\ldots,k$. In the R sector of the mixed strings, the physical vertices are

$$V_\mu = \frac{g_0}{\sqrt{2}}\,\mu^A\,\Delta\,S_A\,\mathrm{e}^{-\frac{1}{2}\phi}\,, \qquad V_{\overline{\mu}} = \frac{g_0}{\sqrt{2}}\,\overline{\mu}^A\,\overline{\Delta}\,S_A\,\mathrm{e}^{-\frac{1}{2}\phi} \tag{2.11}$$

where μ and $\overline{\mu}$ carry the same Chan-Paton factors as w and $\overline{w}$ respectively. Again, it is the GSO projection, together with the conserved supercurrent, that fixes the SO(6) chirality of the spin fields in (2.11).

The vertices described up to now exhaust the BRST invariant spectrum of the open strings with at least one end point on the D-instantons. However, to compute the couplings among the moduli and derive the ADHM measure on moduli space from string interactions, it is convenient to introduce also some *auxiliary moduli* that disentangle quartic interactions [21]. In this context a particularly relevant role is played by the auxiliary vertex

$$V_D = \frac{1}{2}\,D^-_{\mu\nu}\psi^\nu\psi^\mu\,, \tag{2.12}$$

which describes an excitation of the $(-1)/(-1)$ strings associated to an anti-self-dual tensor $D^-_{\mu\nu} = D^-_c\,\overline{\eta}^c_{\mu\nu}$ (where $\overline{\eta}^c_{\mu\nu}$ are the three anti-self-dual 't Hooft symbols).

The transformation properties under the various groups and the scaling dimensions of all ADHM moduli are summarized in table 1.

If we now compute all tree-level diagrams with insertions of the vertices listed above and take the field theory limit $\alpha' \to 0$ (with $g_{\rm YM}$ fixed and hence $g_0 \to \infty$), we obtain the complete ADHM measure for the instanton moduli space of the $\mathcal{N} = 4$ SYM theory (see for instance eq. (3.29) in [21]). An essential point is that the moduli D^-_c and $\lambda_{\dot\alpha A}$ appear in this measure as Lagrange multipliers, respectively, for the bosonic and fermionic ADHM constraints. In particular, the bosonic constraints are the following three $k \times k$ matrix equations

$$W^c + \mathrm{i}\,\overline{\eta}^c_{\mu\nu}\big[a'^\mu, a'^\nu\big] = \mathbf{0}\,, \tag{2.13}$$

where $(W^c)_j{}^i = w_{\dot\alpha}{}^{iu}\,(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\overline{w}^{\dot\beta}{}_{uj}$ in terms of the Pauli matrices τ^c, while the fermionic constraints are

$$w_{\dot\alpha}{}^u\,\overline{\mu}^A_u + \mu^{uA}\overline{w}_{\dot\alpha u} + \big[a'_{\alpha\dot\alpha}, M'^{\alpha A}\big] = \mathbf{0}\,. \tag{2.14}$$

As explained in [21], in the D3/D(−1) system it is possible to consider also disk diagrams with both mixed boundary conditions and insertions of massless vertices of the 3/3 strings associated to gauge fields, and show that on such mixed disks the various components of the gauge multiplet may have non-trivial tadpoles and a non-vanishing space-time profile. For example, for the vector field A_μ one finds indeed that

$$\langle\, \mathcal{V}_{A_\mu} \,\rangle_{\text{mixed disk}} \neq 0 \tag{2.15}$$

where $\mathcal{V}_{A_\mu}$ is the gluon vertex V_A defined in (2.3) without polarization. Furthermore, by taking the Fourier transform of these massless tadpoles, after including a propagator and imposing the ADHM constraints, one obtains [21] a space-time profile which is precisely that of the classical gauge instanton solution in the singular gauge. In other words, the D-instantons act as sources emitting non-abelian gauge fields.

3. D3/D(–1) systems in presence of a B-field

In this section we consider systems of D3 and D(−1) branes in presence of a constant anti-symmetric tensor B of the closed string NS-NS sector, and focus in particular on the massless spectrum of the various kinds of open strings to study their relation with the ADHM instanton construction in non-commutative gauge theories.

The action for superstrings moving in a background B-field is [5]

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int d\sigma\, d\tau \Big\{ \delta_{MN}\, \partial_a X^M \partial^a X^N + \epsilon^{ab} B_{MN}\, \partial_a X^M \partial_b X^N \Big\} + \\ &-\frac{\mathrm{i}}{4\pi}\int d\sigma\, d\tau \Big\{ E_{MN}\, \overline{\psi}^M \partial\!\!\!/\, \psi^N \Big\}\,. \end{aligned} \tag{3.1}$$

where $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$, and $E_{MN} = \delta_{MN} + B_{MN}$. Varying S, we get a bosonic boundary term

$$\int d\tau \Big[\delta X_M \left(\partial_\sigma X^M - B^M{}_N \partial_\tau X^N \right) \Big]_{\sigma=0}^{\sigma=\pi}\,,$$

and a fermionic one

$$\int d\tau \Big[E_{MN} \left(\psi_+^M \delta\psi_+^N - \psi_-^M \delta\psi_-^N \right) \Big]_{\sigma=0}^{\sigma=\pi}$$

where $\psi_\pm^M$ are the left and right-moving components of the world-sheet spinors ψ^M. The boundary terms vanish after imposing boundary conditions of Dirichlet (**D**) or Neumann (**N**) type on the open string fields. For the bosonic coordinates we have

$$\mathbf{D}: \quad \delta X^M \Big|_{\sigma=\bar\sigma} = 0 \qquad \Rightarrow \qquad \partial_\tau X^M \Big|_{\sigma=\bar\sigma} = 0 \tag{3.2}$$

or

$$\mathbf{N}: \quad \left(\partial_\sigma X^M - B^M{}_N \partial_\tau X^N \right) \Big|_{\sigma=\bar\sigma} = 0 \tag{3.3}$$

[5] For the fermionic part we use the action given in [10] which enjoys the property that the boundary terms in its variation can be canceled by consistently imposing on ψ^M and $\delta\psi^M$ the *same* constraints. See for instance [11] for a discussion of the brane supersymmetry in presence of B field within the GS formalism.

where $\bar\sigma = 0$ or $\bar\sigma = \pi$. For the fermionic fields, instead, the presence of B requires some extra care. The Dirichlet boundary conditions are as usual

$$\mathbf{D}: \quad \left(\delta\psi_+^M + \eta_{\bar\sigma}\delta\psi_-^M\right)\Big|_{\sigma=\bar\sigma} = \left(\psi_+^M + \eta_{\bar\sigma}\psi_-^M\right)\Big|_{\sigma=\bar\sigma} = 0 \tag{3.4}$$

where $\eta_{\bar\sigma} = \pm 1$, but there are two inequivalent ways of imposing the Neumann boundary conditions, namely

$$\mathbf{N(a)}: \left(E_{NM}\delta\psi_+^N - \eta_{\bar\sigma}E_{MN}\delta\psi_-^N\right)\Big|_{\sigma=\bar\sigma} = \left(E_{NM}\psi_+^N - \eta_{\bar\sigma}E_{MN}\psi_-^N\right)\Big|_{\sigma=\bar\sigma} = 0 \tag{3.5a}$$

$$\mathbf{N(b)}: \left(E_{MN}\delta\psi_+^N - \eta_{\bar\sigma}E_{NM}\delta\psi_-^N\right)\Big|_{\sigma=\bar\sigma} = \left(E_{MN}\psi_+^N - \eta_{\bar\sigma}E_{NM}\psi_-^N\right)\Big|_{\sigma=\bar\sigma} = 0\,. \tag{3.5b}$$

Clearly, if $B = 0$ there is no distinction between (3.5a) and (3.5b), but if $B \neq 0$ they are different and thus there will be various fermionic sectors when at least one endpoint of the open string has boundary conditions of Neumann type.

In the following we will consider in detail a D3/D(-1) system with a constant background field B along the four world-volume directions of the D3 branes, analyze the different kinds of open strings that are present and make contact with non-commutative field theories and the corresponding ADHM instanton construction.

3.1 The 3/3 strings

In this sector the longitudinal coordinates X^μ and ψ^μ satisfy, respectively, the boundary conditions of Neumann type (3.3) and (3.5) at both endpoints, while the transverse coordinates X^m, ψ^m satisfy, respectively, the Dirichlet boundary conditions (3.2) and (3.4) at both endpoints.

After performing a Wick rotation on the world-sheet ($\tau \to -\mathrm{i}\tau_e$) and introducing the complex variable $z = e^{\tau_e + \mathrm{i}\sigma}$, the bosonic boundary conditions may be written as

$$\partial X^\mu(z,\bar z) = \left(\frac{\mathbb{1}+B}{\mathbb{1}-B}\right)^\mu{}_\nu\, \bar\partial X^\nu(z,\bar z)\,, \tag{3.6a}$$

$$\partial X^m(z,\bar z) = -\,\bar\partial X^m(z,\bar z) \tag{3.6b}$$

for any $z \in \mathbb{R}$. Following [7, 4], we can solve the boundary conditions (3.6) with the doubling trick by introducing holomorphic chiral bosons defined on the entire complex z-plane

$$X^M(z) = q^M - 2\mathrm{i}\alpha' p^M \log z + \mathrm{i}\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}-\{0\}} \frac{\alpha_n^M}{n}\, z^{-n}\,, \tag{3.7}$$

and writing

$$X^\mu(z,\bar z) = \frac{1}{2}\Big[X^\mu(z) + \left(\frac{\mathbb{1}-B}{\mathbb{1}+B}\right)^\mu{}_\nu\, X^\nu(\bar z)\Big]\,, \tag{3.8a}$$

$$X^m(z,\bar z) = x_0^m + \frac{1}{2}\Big[X^m(z) - X^m(\bar z)\Big] \tag{3.8b}$$

for any z with $\mathrm{Im}(z) \geq 0$. In (3.8b) x_0^m denotes the position of the D3 brane in the transverse space, which can be set to zero without loss of generality. Upon canonical quantization the oscillators in (3.7) become operators that satisfy the following commutation relations

$$\big[q^\mu, q^\nu\big] = 2\pi\mathrm{i}\,\alpha' B^{\mu\nu}\,, \quad \big[q^M, p^N\big] = \mathrm{i}\,\delta^{MN}\,, \quad \big[\alpha_n^M, \alpha_n^N\big] = n\,\delta_{n+m,0}\,\delta^{MN}\,. \tag{3.9}$$

The crucial difference with respect to the case at zero background is the non trivial commutator among the longitudinal q's that implies that the geometry on the world-volume of the D3 brane is non-commutative with a non-commutativity parameter [6]

$$\theta^{\mu\nu} = 2\pi\alpha' B^{\mu\nu} \tag{3.10}$$

which is kept fixed in the field theory limit $\alpha' \to 0$.

Let us now consider the fermionic coordinates. As already pointed out, when $B \neq 0$ there are two ways of imposing the boundary conditions of Neumann type on the ψ's and thus, in principle, there are four different fermionic sectors. One possibility is to impose the conditions (3.5a) at both endpoints for the longitudinal directions, i.e.

$$\psi_+^\mu(z) = \eta_0 \left(\frac{1\!\!1 + B}{1\!\!1 - B}\right)^\mu{}_\nu \psi_-^\nu(\overline{z}) \qquad \text{for } z \in \mathbb{R}_+\,, \tag{3.11a}$$

$$\psi_+^\mu(z) = \eta_\pi \left(\frac{1\!\!1 + B}{1\!\!1 - B}\right)^\mu{}_\nu \psi_-^\nu(\overline{z}) \qquad \text{for } z \in \mathbb{R}_-\,, \tag{3.11b}$$

and the conditions (3.4) for the transverse directions, i.e.

$$\psi_+^m(z) = -\eta_0\, \psi_-^m(\overline{z}) \qquad \text{for } z \in \mathbb{R}_+\,, \tag{3.12a}$$

$$\psi_+^m(z) = -\eta_\pi\, \psi_-^m(\overline{z}) \qquad \text{for } z \in \mathbb{R}_-\,. \tag{3.12b}$$

As usual, if $\eta_0\eta_\pi = 1$ we obtain the R sector, while if $\eta_0\eta_\pi = -1$ we obtain the NS sector. The boundary constraints (3.11) and (3.12) can be solved by introducing holomorphic fermionic fields defined on the entire complex plane such that

$$\psi^M(\mathrm{e}^{2\pi\mathrm{i}}\, z) = -\eta_0\eta_\pi\, \psi^M(z)\,, \tag{3.13}$$

and then by writing

$$\psi_+^\mu(z) = z^{\frac{1}{2}}\psi^\mu(z)\,, \quad \psi_-^\mu(\overline{z}) = \eta_0\, \overline{z}^{\frac{1}{2}} \left(\frac{1\!\!1 - B}{1\!\!1 + B}\right)^\mu{}_\nu \psi^\nu(\overline{z})\,, \tag{3.14a}$$

$$\psi_+^m(z) = z^{\frac{1}{2}}\psi^m(z)\,, \quad \psi_-^m(\overline{z}) = -\eta_0\, \overline{z}^{\frac{1}{2}}\, \psi^m(\overline{z}) \tag{3.14b}$$

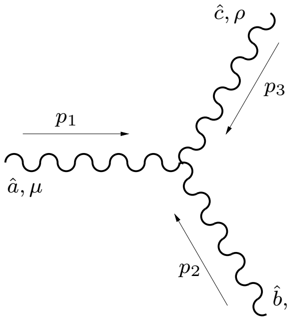
for any z with $\mathrm{Im}(z) \geq 0$. From (3.13) it easily follows that

$$\psi^M(z) = \sum_{r\in\mathbb{Z}+\nu} \psi_r^M\, z^{-r-1/2} \tag{3.15}$$

where $\nu = 0$ in the R sector and $\nu = 1/2$ in the NS sector. Upon canonical quantization the fermionic modes in (3.15) become operators that satisfy the standard anti-commutation relations

$$\left\{\psi_r^M, \psi_s^N\right\} = \delta_{r+s,0}\, \delta^{MN}\,. \tag{3.16}$$

[6]It is worth pointing out that the expression of the open string coordinates written in (3.8a) is different from the one usually considered in the literature. In particular, with our choice the open string metric is equal to the closed string one (i.e. $\delta^{\mu\nu}$ in our case) and the non-commutativity parameter θ is simply proportional to the background field B as shown in (3.10). This is to be contrasted with the Seiberg-Witten approach [8] where a different scaling is considered. A discussion on the relation between these two approaches can be found for example in ref. [13].



$$V^{\mu\nu\rho}_{\hat{a}\hat{b}\hat{c}}(p_1,p_2,p_3) = \left[(p_3-p_2)^\mu\delta^{\nu\rho} + (p_2-p_1)^\rho\delta^{\mu\nu} + (p_1-p_3)^\nu\delta^{\rho\mu}\right] \times \left[f_{\hat{a}\hat{b}\hat{c}}\cos(p_1\wedge p_2) + d_{\hat{a}\hat{b}\hat{c}}\sin(p_1\wedge p_2)\right]$$

Figure 1: The three-gluon vertex in the non-commutative U(N) Yang-Mills theory.

From (3.9) and (3.16) we clearly see that the excitation spectrum of these open strings is isomorphic to the one of the 3/3 strings without the B background. In particular at the massless level we find a gauge field A_μ and six scalars ϕ_m, together with their fermionic partners $\Lambda^{\alpha A}$ and $\overline{\Lambda}_{\dot{\alpha}A}$ that complete a $\mathcal{N}=4$ vector supermultiplet in the adjoint representation of U(N). The corresponding vertex operators have the same expressions as those listed in section 2, see in particular eqs. (2.3) and (2.4). However, since the longitudinal zero-modes q^μ's contained in the exponential $\mathrm{e}^{\mathrm{i}p\cdot X}$ satisfy non-trivial commutation relations, the scattering amplitudes among these vertices are modified and the resulting gauge theory becomes non-commutative [8]. Typically, in the field theory limit where the non-commutative parameter θ defined in (3.10) is kept fixed, various interaction terms may acquire momentum factors like $\cos(p_1\wedge p_2)$ and $\sin(p_1\wedge p_2)$ where

$$p_1\wedge p_2 = \frac{1}{2}\,p_1^\mu\,\theta_{\mu\nu}\,p_2^\nu\,. \tag{3.17}$$

Furthermore, new structures may appear as well (see for instance ref. [38, 39]). For instance in the 3-gluon vertex the usual term proportional to the structure constants of U(N) is modified with $\cos(p_i\wedge p_j)$ factors and a term proportional to the $d_{\hat{a}\hat{b}\hat{c}}$ tensor shows up in the non-commutative case (see figure 1).

However, there are other ways of imposing the Neumann boundary conditions on the fermionic fields. For example, we could require that both endpoints of the open string satisfy conditions of type (3.5b). In this case, essentially nothing changes with respect to what discussed above. In fact, to solve these boundary conditions one still introduces chiral fermions $\psi'^M(z)$ with the same monodromy properties, and hence the same mode expansion, of the fields $\psi^M(z)$ defined in (3.15). Therefore, the resulting spectrum is simply a copy of the one previously considered, and in particular at the massless level we find a gauge vector multiplet. If instead we impose the boundary conditions (3.5a) at $\sigma=0$ and the conditions (3.5b) at $\sigma=\pi$, or vice-versa, things are radically different. In fact, to solve the corresponding constraints we have to introduce chiral fermions $\chi^\mu(z)$ such that

$$\chi^\mu(\mathrm{e}^{2\pi\mathrm{i}}\,z) = -\eta_0\eta_\pi\Big[\Big(\frac{\mathbb{1}\pm B}{\mathbb{1}\mp B}\Big)^2\Big]^\mu{}_\nu\,\chi^\nu(z)\,.$$

These fields are no longer periodic or anti-periodic, and hence their modes are no longer integers or half-integers. Moreover, it can be checked that the physical spectrum constructed using these modes does not contain massless states even in the field theory limit.

In particular it is not possible to obtain a massless gauge vector with this "mixed" choice of fermionic boundary conditions. Therefore, for our purposes such strings do not play any role and can be consistently neglected in the classical approximation.

In conclusions, the 3/3 strings have only two sectors that in the field theory limit reproduce a non-commutative $\mathcal{N} = 4$ SYM theory. To just describe this gauge theory it would be sufficient to consider only one of them, as is usually done in the literature. However, as we shall see later, to obtain also the non-commutative ADHM instanton construction from string theory it is necessary to consider both sectors in a symmetric way, i.e. identify states with equal quantum numbers. This implies, for instance, that the gluon emission vertex in the (-1) superghost picture is

$$V_A(p) = \frac{A_\mu(p)}{\sqrt{2}} \, \frac{\psi^\mu + \psi'^\mu}{\sqrt{2}} \, \mathrm{e}^{-\phi} \, \mathrm{e}^{\mathrm{i} p\cdot X} \, . \tag{3.18}$$

However, in all practical calculations we can simply identify ψ^μ and ψ'^μ, and still use the properly normalized gluon vertex (2.3) and the standard contraction rules.

3.2 The (–1)/(–1) strings

The open strings with both endpoints on the D(-1) branes have Dirichlet boundary conditions (3.2) and (3.4) in all directions and hence do not feel any effect of the B background. All bosonic coordinates X^M have an expansion like (3.8b) with x_0^M denoting the position of the D-instantons, while all fermionic coordinates ψ^M are as in (3.14b). The physical spectrum of these (–1)/(–1) strings contains the same states as in the free case, describing the moduli a'_μ, χ_m, $M'^{\alpha A}$ and $\lambda_{\dot\alpha A}$ together with the auxiliary fields D_c^-. Their corresponding vertices have the same expressions as in (2.7), (2.8) and (2.12).

3.3 The (–1)/3 and 3/(–1) strings

We now consider the open strings that stretch between a D(-1) and a D3 brane. To simplify our discussion, but without loosing generality, we assume that the background field $B_{\mu\nu}$ is in the skew-diagonal form

$$B = \begin{pmatrix} \begin{matrix} 0 & b^{\dot 2} \\ -b^{\dot 2} & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & b^{\dot 1} \\ -b^{\dot 1} & 0 \end{matrix} \end{pmatrix} , \tag{3.19}$$

so that it becomes natural to introduce the complex fields

$$Z^{\dot 1} = \frac{X^3 + \mathrm{i}\, X^4}{\sqrt{2}} \, , \qquad Z^{\dot 2} = \frac{X^1 + \mathrm{i}\, X^2}{\sqrt{2}} \, , \tag{3.20a}$$

$$\Psi^{\dot 1} = \frac{\psi^3 + \mathrm{i}\, \psi^4}{\sqrt{2}} \, , \qquad \Psi^{\dot 2} = \frac{\psi^1 + \mathrm{i}\, \psi^2}{\sqrt{2}} \, . \tag{3.20b}$$

As we will see in the following, the use of dotted indices, like for anti-chiral spinors, turns out to be particularly useful. In the above complex basis the bosonic boundary conditions

JHEP05(2006)069

for the longitudinal coordinates of a $(-1)/3$ string become

$$\partial Z^{\dot\alpha}(z,\overline{z}) = -\overline{\partial} Z^{\dot\alpha}(z,\overline{z}) \qquad \text{for } z \in \mathbb{R}_+ \,, \tag{3.21a}$$

$$\partial Z^{\dot\alpha}(z,\overline{z}) = \left(\frac{1-\mathrm{i}b^{\dot\alpha}}{1+\mathrm{i}b^{\dot\alpha}}\right) \overline{\partial} Z^{\dot\alpha}(z,\overline{z}) \qquad \text{for } z \in \mathbb{R}_- \,, \tag{3.21b}$$

where $\dot\alpha = \dot1, \dot2$. Since the boundary conditions (3.21) are diagonal in each complex direction, i.e. for a given value of $\dot\alpha$, for simplicity we temporarily suppress this index and reinstate it only when necessary.

To solve (3.21) we use again the doubling trick: we introduce a complex chiral field $Z(z)$ such that

$$Z(\mathrm{e}^{2\pi\mathrm{i}}\, z) = -\left(\frac{1-\mathrm{i}b}{1+\mathrm{i}b}\right) Z(z) = -\mathrm{e}^{-2\pi\mathrm{i}\epsilon}\, Z(z) \tag{3.22}$$

where

$$\epsilon = \frac{1}{\pi}\arctan b \qquad \left(-\frac{1}{2} < \epsilon < \frac{1}{2}\right), \tag{3.23}$$

and write

$$Z(z,\overline{z}) = \frac{1}{2}\Big[Z(z) - \overline{Z}(\overline{z})\Big] \tag{3.24}$$

for any z with $\mathrm{Im}(z) \geq 0$. From (3.22) we deduce that

$$Z(z) = \sum_{n\in\mathbb{Z}} \frac{1}{n+\epsilon+\frac{1}{2}}\, \alpha_{n+\epsilon+\frac{1}{2}}\, z^{-n-\epsilon-\frac{1}{2}} \,, \tag{3.25}$$

which, for vanishing background, reduces to the standard half-integer mode expansion of a boson with mixed Dirichlet-Neumann boundary conditions. Canonical quantization leads to the commutators

$$\Big[\alpha_{n+\epsilon+\frac{1}{2}}, \overline{\alpha}_{-m-\epsilon-\frac{1}{2}}\Big] = \Big(n+\epsilon+\frac{1}{2}\Big)\, \delta_{n,m} \tag{3.26}$$

where $\overline{\alpha}_{-m-\epsilon-\frac{1}{2}}$ are the modes that appear in the expansion of the complex conjugate field $\overline{Z}$. The oscillators with positive index are annihilation operators with respect to the twisted vacuum $|\epsilon\rangle$, namely

$$\alpha_{n+\epsilon+\frac{1}{2}}|\epsilon\rangle = 0 \quad (n \geq 0) \qquad \text{and} \qquad \overline{\alpha}_{n-\epsilon-\frac{1}{2}}|\epsilon\rangle = 0 \quad (n \geq 1)\,,$$

whereas the modes with negative index are creation operators. The contribution of this twisted boson to the Virasoro generator L_0 is

$$L_0^{(Z)} = \sum_{n\in\mathbb{Z}} :\overline{\alpha}_{-n-\epsilon-\frac{1}{2}}\alpha_{n+\epsilon+\frac{1}{2}}: + \left(\frac{1}{8} - \frac{\epsilon^2}{2}\right)$$

where the normal ordering is defined with respect to the twisted vacuum introduced above. Thus, $|\epsilon\rangle$ has conformal dimension $h = 1/8 - \epsilon^2/2$ and is created from the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum $|0\rangle$ by a twist field $\sigma(z)$ of weight h, namely

$$|\epsilon\rangle = \lim_{z\to 0} \sigma(z)|0\rangle \,.$$

Let us now consider the longitudinal fermionic coordinates (3.20b), for which the boundary conditions are

$$\Psi_+^{\dot\alpha}(z) = -\eta_0 \Psi_-^{\dot\alpha}(\overline{z}) \qquad \text{for } z \in \mathbb{R}_+ \,, \tag{3.27a}$$

$$\Psi_+^{\dot\alpha}(z) = \eta_\pi \left(\frac{1-\mathrm{i}b^{\dot\alpha}}{1+\mathrm{i}b^{\dot\alpha}}\right) \Psi_-^{\dot\alpha}(\overline{z}) \qquad \text{for } z \in \mathbb{R}_- \tag{3.27b}$$

if we choose the form (3.5a) of the Neumann relation at $\sigma = \pi$. As before, we solve these constraints using the doubling trick: for each value of the index $\dot\alpha$ we introduce a multi-valued chiral fermion $\Psi(z)$ such that

$$\Psi(\mathrm{e}^{2\pi\mathrm{i}}\, z) = \eta_0\eta_\pi\, \mathrm{e}^{-2\pi\mathrm{i}\epsilon}\, \Psi(z) \tag{3.28}$$

and write

$$\Psi_+(z) = z^{\frac{1}{2}}\Psi(z)\,, \qquad \Psi_-(\overline{z}) = -\eta_0\, \overline{z}^{\frac{1}{2}}\, \Psi(\overline{z}) \tag{3.29}$$

for any z with $\mathrm{Im}(z) \geq 0$. From the monodromy property (3.28) we easily find that

$$\Psi(z) = \sum_{n\in\mathbb{Z}+\nu} \Psi_{n+\epsilon} z^{-n-\epsilon-\frac{1}{2}} \tag{3.30}$$

where $\nu = 0$ in the NS sector ($\eta_0\eta_\pi = -1$) and $\nu = 1/2$ in the R sector ($\eta_0\eta_\pi = 1$). Canonical Dirac quantization leads to the following non-vanishing anti-commutators

$$\left\{\Psi_{n+\epsilon}, \overline{\Psi}_{-m-\epsilon}\right\} = \delta_{n,m} \tag{3.31}$$

where $\overline{\Psi}_{-m-\epsilon}$ are the modes of the complex conjugate field $\overline{\Psi}$. Notice that in the presence of a B field, neither the NS nor the R sectors of the mixed directions have zero-modes, and thus for the $(-1)/3$ strings the twisted fermionic vacuum $|\epsilon\rangle$, annihilated by all positive modes, is always non degenerate. The contribution of Ψ to the Virasoro operator L_0 is

$$L_0^{(\Psi)} = \sum_{n\in\mathbb{Z}+\nu} (n+\epsilon)\ :\overline{\Psi}_{-n-\epsilon}\Psi_{n+\epsilon}:\ + a_\nu \tag{3.32}$$

where the normal ordering constant is

$$a_0 = \frac{1}{2}\left(\frac{1}{2} - |\epsilon|\right)^2\,, \qquad a_{\frac{1}{2}} = \frac{\epsilon^2}{2} \tag{3.33}$$

in the NS and R sectors respectively.

The twisted vacuum of the NS sector $|\epsilon\rangle_{\rm NS}$, whose energy is a_0, is created from the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum $|0\rangle$ by the spin-twist field $s^+(z)$ when $\epsilon > 0$ and by $s^-(z)$ when $\epsilon < 0$, namely

$$|\epsilon\rangle_{\rm NS} = \begin{cases} \lim\limits_{z\to 0} s^+(z)|0\rangle & \text{for } \epsilon > 0\,, \\ \lim\limits_{z\to 0} s^-(z)|0\rangle & \text{for } \epsilon < 0\,. \end{cases}$$

The spin-twist fields $s^\pm$ are most easily described in the bosonization formalism where

$$\Psi(z) = \mathrm{e}^{+\mathrm{i}\varphi(z)}\,, \qquad \overline{\Psi}(z) = \mathrm{e}^{-\mathrm{i}\varphi(z)} \tag{3.34}$$

up to cocycle terms.[7] Then, one can show that

$$s^{\pm}(z) = \mathrm{e}^{\pm\mathrm{i}\left(\frac{1}{2}-|\epsilon|\right)\varphi(z)} . \tag{3.35}$$

In the following we will need to consider also the first excited state of the NS sector. If $\epsilon > 0$, this is

$$\overline{\Psi}_{-\epsilon}|\epsilon\rangle_{\rm NS} = \lim_{z\to 0} \overline{t}^{+}(z)|0\rangle_{\rm NS}$$

where $\overline{t}^{+}(z)$ is an excited spin-twist field defined by the OPE

$$\overline{\Psi}(z)s^{+}(w) = \frac{\overline{t}^{+}(w)}{(z-w)^{\frac{1}{2}-\epsilon}} + \cdots .$$

In the bosonization formalism, this OPE allows us to write $\overline{t}^{+}(z) = \mathrm{e}^{-\mathrm{i}(\frac{1}{2}+\epsilon)\varphi(z)}$ whose conformal dimension is $a_0 + \epsilon = \frac{1}{2}(\frac{1}{2}+\epsilon)^2$. For $\epsilon < 0$, instead, the first excited state is

$$\Psi_{\epsilon}|\epsilon\rangle_{\rm NS} = \lim_{z\to 0}\, t^{-}(z)|0\rangle$$

where $t^{-}(z) = \mathrm{e}^{\mathrm{i}(\frac{1}{2}-\epsilon)\varphi(z)}$ whose conformal dimension is $a_0 - \epsilon = \frac{1}{2}(\frac{1}{2}-\epsilon)^2$.

In the R sector, the twisted vacuum $|\epsilon\rangle_{\rm R}$ is created from the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum by the spin-twist field

$$s_{\rm R}(z) = e^{-\mathrm{i}\epsilon\varphi(z)}$$

whose conformal dimension is $\epsilon^2/2$. For our future applications we will not need to consider excited states in the R sector, but of course they could be easily constructed along the same lined discussed for the NS case.

It is important to realize that if we had chosen the other type of Neumann boundary conditions for the longitudinal fermionic coordinates, i.e. (3.5b), we would have retrieved the same expressions as above at all stages, but with $b \to -b$, or equivalently with $\epsilon \to -\epsilon$. Thus, we can conclude that for the $(-1)/3$ strings, the two possible choices of fermionic Neumann boundary conditions are simply related to each other by the exchange of Ψ and $\overline{\Psi}$.

3.3.1 Spectrum

Let us now discuss the physical spectrum of the mixed strings. Due to the absence of momentum in all directions, there are very severe constraints on the form of allowed states and only very few of them are physical.

In the NS sector the twisted vacuum $|\epsilon_1;\epsilon_2\rangle_{\rm NS}$ cannot be physical. Let us see why. If, for example, $\epsilon_1, \epsilon_2 > 0$, the vacuum is described by the following vertex operator in the (-1)-superghost picture

$$\sigma_1\, s_1^{+}\, \sigma_2\, s_2^{+}\, \mathrm{e}^{-\phi} \quad ;$$

if ϵ_1 or ϵ_2 are negative, the corresponding twist fields s^+ must be replaced by s^-. The conformal dimension of any of these vertices is $h = 1-(|\epsilon_1|+|\epsilon_2|)/2$. Thus, h can never be 1

[7] As explained in the appendix, complex conjugation acts as $\left(\Psi^{\dot\alpha}\right)^* = \overline{\Psi}_{\dot\alpha}$.

in a non-trivial background. Physical states can instead be present in the first excited level. When the excitation is produced by the longitudinal fermions, there are these possibilities

$$\overline{\Psi}_{\dot{1}\,,-\epsilon_1}|\epsilon_1;\epsilon_2\rangle_{\rm NS}\,,\quad \Psi^{\dot{1}}_{\epsilon_1}|\epsilon_1,\epsilon_2\rangle_{\rm NS}\,,\quad \overline{\Psi}_{\dot{2}\,,-\epsilon_2}|\epsilon_1;\epsilon_2\rangle_{\rm NS}\,,\quad \Psi^{\dot{2}}_{\epsilon_2}|\epsilon_1,\epsilon_2\rangle_{\rm NS}$$

depending on whether $\epsilon_1 > 0$, $\epsilon_1 < 0$, $\epsilon_2 > 0$ or $\epsilon_2 < 0$. If, for example, $\epsilon_1, \epsilon_2 > 0$, we have the following two vertex operators (in the (-1) superghost picture)

$$\mathcal{V}_{\dot{1}} = \sigma_1\, \overline{t}_1^{+}\, \sigma_2\, s_2^{+}\, \mathrm{e}^{-\phi} \quad \text{and} \quad \mathcal{V}_{\dot{2}} = \sigma_1\, s_1^{+}\, \sigma_2\, \overline{t}_2^{+}\, \mathrm{e}^{-\phi}\,. \tag{3.36}$$

Again, if either ϵ_1 or ϵ_2 are negative, we must replace s^+ and t^+ with s^- and t^- in the appropriate places. The total conformal weight of these vertices is

$$h\,(\mathcal{V}_{\dot{1}}) = 1 + \frac{|\epsilon_1| - |\epsilon_2|}{2}\,, \qquad h\,(\mathcal{V}_{\dot{2}}) = 1 - \frac{|\epsilon_1| - |\epsilon_2|}{2}$$

and thus they are physical only if $|\epsilon_1| = |\epsilon_2|$, i.e. when the background is self-dual or anti-self-dual. Extending this analysis, one can easily prove that no other physical states exist in the NS sector.

In the R sector the only physical state turns out to be the vacuum, which carries indices of the spinor representation of SO(6) due to the zero-modes of the ψ^m fields in the transverse directions. This vacuum is thus associated to the transverse spin fields S_A or S^A, and the corresponding vertex operators in the $(-1/2)$ superghost picture are

$$\mathcal{V}_A = \sigma_1\, s_{R,1}\, \sigma_2\, s_{R,2}\, S_A\, \mathrm{e}^{-\frac{1}{2}\phi} \quad \text{and} \quad \mathcal{V}^A = \sigma_1\, s_{R,1}\, \sigma_2\, s_{R,2}\, S^A\, \mathrm{e}^{-\frac{1}{2}\phi} \tag{3.37}$$

which have conformal weight 1, because the ϵ contributions cancel between the bosonic and the fermionic terms. Thus, the vertices (3.37) describe physical states. No other (excited) state of the R sector is physical.

3.3.2 Moduli spectrum in (anti-)self-dual background

The previous discussion shows that the physical NS sector is non-empty only when the background has a definite duality. For definiteness, let us consider a self-dual B field with $\epsilon_1 = \epsilon_2 = \epsilon > 0$. In this case the physical NS vertices are given by (3.36) and in the bosonized formalism their fermionic parts read

$$\mathcal{V}_{\dot{1}} \sim \mathrm{e}^{-\frac{\mathrm{i}}{2}(\varphi_1-\varphi_2)-\mathrm{i}\epsilon(\varphi_1+\varphi_2)} \quad \text{and} \quad \mathcal{V}_{\dot{2}} \sim \mathrm{e}^{+\frac{\mathrm{i}}{2}(\varphi_1-\varphi_2)-\mathrm{i}\epsilon(\varphi_1+\varphi_2)}\,. \tag{3.38}$$

We denote collectively these vertices by $\mathcal{V}_{\dot{\alpha}}$ with $\dot{\alpha} = \dot{1}, \dot{2}$, since they are created by the action of $\overline{\Psi}_{\dot{\alpha}\,,-\epsilon}$ on the twisted vacuum. The label $\dot{\alpha}$ suggests that they transform as an anti-chiral spinor of SO(4). To prove this, let us write the SO(4) generators as

$$J_{\mu\nu} \equiv\, :\psi_\mu\psi_\nu:\, =\; \eta^c_{\mu\nu} J^{(+)}_c + \overline{\eta}^c_{\mu\nu} J^{(-)}_c \tag{3.39}$$

where $J^{(+)}_c$ and $J^{(-)}_c$ are the $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$ currents respectively, and then use the bosonized formalism to get (up to cocycles)

$$J^{(+)}_3 = \frac{1}{2}\left(\partial\varphi_1 + \partial\varphi_2\right), \quad J^{(+)}_\pm = \mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}(\varphi_1+\varphi_2)}\,, \tag{3.40a}$$

$$J^{(-)}_3 = \frac{1}{2}\left(\partial\varphi_1 - \partial\varphi_2\right), \quad J^{(-)}_\pm = \mathrm{i}\,\mathrm{e}^{\mp\mathrm{i}(\varphi_1-\varphi_2)}\,, \tag{3.40b}$$

where $J_\pm^{(\pm)} = \big(J_1^{(\pm)} \pm \mathrm{i} J_2^{(\pm)}\big)$ are the step operators of the $\mathrm{SU}(2)_\pm$ groups. In a self-dual skew-diagonal background field $B_{\mu\nu}$, the Lorentz group is broken to $\mathrm{U}(1)_+ \times \mathrm{SU}(2)_-$, where $\mathrm{U}(1)_+$ is the subgroup of $\mathrm{SU}(2)_+$ generated by $J_3^{(+)}$. Using (3.38) and (3.40), it is elementary to obtain

$$J_c^{(-)}(z)\, \mathcal{V}_{\dot\alpha}(w) = \frac{\mathrm{i}}{2}\, \frac{\mathcal{V}_{\dot\beta}(w)\, (\tau_c)^{\dot\beta}{}_{\dot\alpha}}{z-w} + \cdots \tag{3.41}$$

which shows that indeed the vertices $\mathcal{V}^{\dot\alpha}$ transform as a doublet of $\mathrm{SU}(2)_-$, and

$$J_3^{(+)}(z)\, \mathcal{V}_{\dot\alpha}(w) = \mathrm{i}\, \frac{\epsilon\, \mathcal{V}_{\dot\alpha}(w)}{z-w} + \cdots \tag{3.42}$$

which shows that these vertices have charge ϵ under $\mathrm{U}(1)_+$ [8]. Due to this non-zero charge, the vertices $\mathcal{V}_{\dot\alpha}$ are *not* true anti-chiral spinors and, contrarily to naive expectations, they *cannot* be associated with the moduli $w_{\dot\alpha}$ of the ADHM construction, which, as shown in table 1, are singlets under $\mathrm{SU}(2)_+$ and hence carry zero charge under $\mathrm{U}(1)_+$. Notice that when $B = 0$ the physical NS vertex operators (2.10) have the quantum numbers of the degenerate twisted NS vacuum; on the contrary when $B \neq 0$ the physical NS vertices (3.36) are associated to the first fermionic excited states on a non-degenerate scalar vacuum and hence carry the quantum numbers of the fermionic oscillators, which are Lorentz vectors. This explains the origin of the non-vanishing charge of $\mathcal{V}_{\dot\alpha}$ under $\mathrm{U}(1)_+$.

This problem can be overcome thanks to the existence of another way of realizing the $(-1)/3$ strings. So far, in fact, we have used fermionic fields $\Psi^{\dot\alpha}$ and $\overline{\Psi}_{\dot\alpha}$ that satisfy the Neumann boundary conditions of type **N(a)** (see eq. (3.27b)). However, also the boundary conditions of type **N(b)** can be used. With this second choice, everything goes formally as before except that in the fermionic sector ϵ is everywhere replaced by $-\epsilon$ and the roles of $\Psi^{\dot\alpha}$ and $\overline{\Psi}_{\dot\alpha}$ are exchanged. Thus, in the new NS sector, for a self-dual background with $\epsilon > 0$, the physical vertex operators are

$$\mathcal{V}'^{\dot 1} = \sigma_1\, t_1^-\, \sigma_2\, s_2^-\, \mathrm{e}^{-\phi} \quad \text{and} \quad \mathcal{V}'^{\dot 2} = \sigma_1\, s_1^-\, \sigma_2\, t_2^-\, \mathrm{e}^{-\phi} \tag{3.43}$$

instead of the ones given in (3.36). Computing their OPE's with the preserved Lorentz generators one finds

$$J_c^{(-)}(z)\, \mathcal{V}'_{\dot\alpha}(w) = \frac{\mathrm{i}}{2}\, \frac{\mathcal{V}'_{\dot\beta}(w) (\tau_c)^{\dot\beta}{}_{\dot\alpha}}{z-w} + \cdots\,, \tag{3.44a}$$

$$J_3^{(+)}(z)\, \mathcal{V}'^{\dot\alpha}(w) = -\,\mathrm{i}\, \frac{\epsilon\, \mathcal{V}'^{\dot\alpha}(w)}{z-w} + \cdots\,, \tag{3.44b}$$

which show that the new vertices form again a doublet of $\mathrm{SU}(2)_-$ but carry opposite $\mathrm{U}(1)_+$ charge with respect to the old vertices $\mathcal{V}$.

In complete analogy with what we did on the 3/3 strings, and in order to be consistent with that choice, also here we treat the two types of boundary conditions for the $(-1)/3$ strings in a symmetric way, and thus consider the following projected vertex operator

$$V_w = \frac{g_0}{\sqrt{2}}\, w_{\dot\alpha}\, \frac{\mathcal{V}^{\dot\alpha} + \mathcal{V}'^{\dot\alpha}}{\sqrt{2}}\,, \tag{3.45}$$

[8] Correctly, no simple poles appear in the OPE of $\mathcal{V}^{\dot\alpha}$ with the broken generators $J_\pm^{(+)}$ of $\mathrm{SU}(2)_+$.

JHEP05(2006)069

	$\mathrm{U}(1)_+$	$\mathrm{SU}(2)_-$
$\mathcal{V}^{\dot\alpha}$	ϵ	$\mathbf{2}$
$\mathcal{V}'^{\dot\alpha}$	$-\epsilon$	$\mathbf{2}$

	$\mathrm{SU}(2)_+$	$\mathrm{U}(1)_-$
$\mathcal{V}^{\alpha}$	$\mathbf{2}$	ϵ
$\mathcal{V}'^{\alpha}$	$\mathbf{2}$	$-\epsilon$

Table 2: Transformation properties of the physical NS vertices under the unbroken part of the Lorentz group in a self-dual ($\epsilon_1 = \epsilon_2 = \epsilon$) or anti-self-dual ($\epsilon_1 = -\epsilon_2 = \epsilon$) background.

where we have inserted a polarization $w_{\dot\alpha}$ and a normalization $g_0/\sqrt{2}$. In this case, however, differently from what we did for the vertices of the 3/3 strings, we cannot simply identify $\mathcal{V}$ and $\mathcal{V}'$, since they have different quantum numbers. As we will see in the following sections, the vertex (3.45) correctly describes the moduli $w_{\dot\alpha}$ of the ADHM construction for the non-commutative gauge theory, and represents the generalization of the vertex (2.10) when a self-dual $B_{\mu\nu}$ background is present.

A few remarks are in order at this point. The projected vertex $\big(\mathcal{V}+\mathcal{V}'\big)$ is a doublet of $\mathrm{SU}(2)_-$, but it clearly does not have a definite $\mathrm{U}(1)_+$ charge. Notice however that in any disk amplitude such a vertex must always be accompanied by its conjugate for consistency of the Chan-Paton structure, and hence all relevant quantities of the ADHM construction involving $w_{\dot\alpha}$ (like constraints, explicit expression of the instanton solution, ...) are actually sensible only to the *expectation value* of $J_3^{(+)}$ between projected states, which indeed vanishes. In other words the polarization appearing in (3.45) has effectively the correct quantum numbers of the ADHM moduli $w_{\dot\alpha}$. We will see an explicit example of this fact in section 4.

This analysis can be easily repeated in the R sector of the mixed string. Here one finds a physical GSO projected and symmetrized vertex given by

$$V_\mu = \frac{g_0}{\sqrt{2}}\,\mu^A\,\frac{\mathcal{V}_A + \mathcal{V}'_A}{\sqrt{2}}\,, \tag{3.46}$$

where $\mathcal{V}_A$ is defined in (3.37) and $\mathcal{V}'_A$ is its analogue with the $\mathbf{N(b)}$ boundary conditions. The vertex (3.46) correctly describes the fermionic ADHM moduli μ^A in the non-commutative gauge theory.

We conclude by mentioning that the $3/(-1)$ strings can be treated in the same way with a simple exchange of the boundary conditions at $\sigma = 0$ and $\sigma = \pi$, and that the ADHM moduli $\overline{w}_{\dot\alpha}$ and $\overline{\mu}^A$ are described by the conjugates of the vertices (3.45) and (3.46).

Finally, if the $B_{\mu\nu}$ background is anti-self-dual and hence the Lorentz group is broken to $\mathrm{SU}(2)_+ \times \mathrm{U}(1)_-$, the physical vertex operators of the NS sector turn out to be doublets of $\mathrm{SU}(2)_+$ with charge under $\mathrm{U}(1)_-$ and can be used to describe the ADHM moduli w_α and $\overline{w}_\alpha$ of non-commutative anti-instantons. The transformation properties and charges of the physical NS vertices are summarized in table 2.

JHEP05(2006)069

3.3.3 The issue of stability

The previous discussion shows that in a self-dual B background the NS sector of the mixed strings contains only the moduli $w_{\dot\alpha}$ and $\overline{w}_{\dot\alpha}$ associated to an instanton, while in an anti-self-dual background it contains only the moduli w_α and $\overline{w}_\alpha$ associated to an anti-instanton. In other words, the explicit string realization of the ADHM construction seems to be applicable only to configurations where the non-commutative gauge field strength and the B background have the *same* duality properties.

We now explain the origin and the physical meaning of this fact. Let us first recall that instantons and anti-instantons are realized respectively by systems of D3/D(-1) branes and systems of D3/anti-D(-1) branes, and that the corresponding mixed strings are characterized by a different GSO projection. Indeed,

$$\mathcal{P}_{\rm GSO} = \frac{1 \pm (-1)^F}{2} \tag{3.47}$$

where the $+$ sign applies to the D3/D(-1) system, and the $-$ sign to the D3/anti-D(-1) system. It is therefore obvious that if a state survives the GSO projection of the D3/D(-1) system, this same state is removed by the GSO projection of the D3/anti-D(-1) system, and vice-versa. Let us consider the NS sector and fix our conventions in such a way that

$$(-1)^F\,|\epsilon_1,\epsilon_2\rangle_{\rm NS} = -\,|\epsilon_1,\epsilon_2\rangle_{\rm NS} \tag{3.48}$$

for $\epsilon_1,\epsilon_2 > 0$. With this choice, the excited states $\overline{\Psi}_{\dot 1\,,-\epsilon_1}|\epsilon_1;\epsilon_2\rangle_{\rm NS}$ and $\overline{\Psi}_{\dot 2\,,-\epsilon_2}|\epsilon_1;\epsilon_2\rangle_{\rm NS}$, which are physical for $\epsilon_1=\epsilon_2$, are selected by the GSO projection of the D3/D(-1) branes. If we follow $|\epsilon_1,\epsilon_2\rangle_{\rm NS}$ (which is annihilated by all positive modes and in particular by $\Psi^{\dot 2}_{\epsilon_2}$) when, say, ϵ_2 decreases and becomes negative, we find that it is mapped to the state $\Psi^{\dot 2}_{\epsilon_2}|\epsilon_1,\epsilon_2\rangle_{\rm NS}$; thus, the very same definition of $(-1)^F$ we used in (3.48), leads to

$$(-1)^F\,|\epsilon_1,\epsilon_2\rangle_{\rm NS} = +\,|\epsilon_1,\epsilon_2\rangle_{\rm NS} \tag{3.49}$$

for $\epsilon_1>0$ and $\epsilon_2<0$. If we now let ϵ_1 become negative as well, the F parity of the vacuum returns to be (-1) as in (3.48). We can then conclude that if ϵ_1 and ϵ_2 have the same sign, the vacuum $|\epsilon_1,\epsilon_2\rangle_{\rm NS}$ has F-parity (-1), whereas if ϵ_1 and ϵ_2 have different signs, the vacuum $|\epsilon_1,\epsilon_2\rangle_{\rm NS}$ has F-parity $(+1)$. Thus, in a self-dual background the physical states, which are forced to stay at the first excited level of the NS sector, survive the GSO projection appropriate for instantons while in an anti-self-dual background they survive the GSO projection of anti-instantons.

This asymmetry has a deep physical meaning: indeed, it is related to the fact that D3/D(-1) systems are stable only in a self-dual background, while D3/anti-D(-1) systems are stable only in an anti-self-dual background [9]. To investigate the stability of these systems, we compute the one-loop free energy for the oriented open strings stretching between the D3 branes and the (anti-)D-instantons in presence of a B background. This free energy is given by

$$\mathcal{F} = -\int_0^{{\rm i}\infty}\frac{d\tau}{2\tau}\left[{\rm Tr}_{\rm NS}\;q^{L_0-\frac{c}{24}} \pm {\rm Tr}_{\rm NS}\;(-1)^F q^{L_0-\frac{c}{24}} - {\rm Tr}_{\rm R}\;q^{L_0-\frac{c}{24}}\right] \tag{3.50}$$

[9] A discussion on related issues appears also in ref.[8].

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ and c is the total central charge of the CFT in the light-cone. Notice that in (3.50) we have not written the trace with $(-1)^F$ in the R sector since it vanishes due to the fermionic zero-modes in the six transverse directions. According to (3.47), the $+$ sign in $\mathcal{F}$ refers to the D3/D(-1) case, while the $-$ sign refers to the D3/anti-D(-1) case.

Let us discuss the contribution to the traces from the four mixed directions which feel the effect of the B background. Actually, we can focus just on the fermionic piece, since all the bosonic contributions to the integrand of (3.50) are common to the various sectors. In computing the traces for the CFT of the twisted fermions we have to take into account in a symmetric way the two types of boundary conditions, $\mathbf{N(a)}$ and $\mathbf{N(b)}$, which they can satisfy (see eqs. (3.5a) and (3.5b)). In practice, this means that

$$\mathrm{Tr}_{\,\mathrm{NS,R}} \to \frac{\mathrm{Tr}^{\,\mathbf{(a)}}_{\,\mathrm{NS,R}} + \mathrm{Tr}^{\,\mathbf{(b)}}_{\,\mathrm{NS,R}}}{2} \,. \tag{3.51}$$

Using (3.32) for the two complex fermions Ψ^1 and Ψ^2 and introducing the standard Jacobi θ-functions, for any value of ϵ_1 and ϵ_2 we find

$$\mathrm{Tr}^{\,\mathbf{(a)}}_{\,\mathrm{NS}}\, q^{L_0 - \frac{c}{24}} = q^{\frac{\epsilon_1^2+\epsilon_2^2}{2}}\, \frac{\theta_2\left(\epsilon_1\tau|\tau\right)\theta_2\left(\epsilon_2\tau|\tau\right)}{\eta(\tau)^2}\,, \tag{3.52a}$$

$$\mathrm{Tr}^{\,\mathbf{(a)}}_{\,\mathrm{NS}}\, (-1)^F\, q^{L_0 - \frac{c}{24}} = q^{\frac{\epsilon_1^2+\epsilon_2^2}{2}}\, \frac{\theta_1\left(\epsilon_1\tau|\tau\right)\theta_1\left(\epsilon_2\tau|\tau\right)}{\eta(\tau)^2}\,, \tag{3.52b}$$

$$\mathrm{Tr}^{\,\mathbf{(a)}}_{\,\mathrm{R}}\, q^{L_0 - \frac{c}{24}} = q^{\frac{\epsilon_1^2+\epsilon_2^2}{2}}\, \frac{\theta_3\left(\epsilon_1\tau|\tau\right)\theta_3\left(\epsilon_2\tau|\tau\right)}{\eta(\tau)^2} \tag{3.52c}$$

where η is the Dedekind's function. The partition functions for the boundary conditions of type $\mathbf{N(b)}$ can be simply obtained from (3.52) by reversing the signs of both ϵ_1 and ϵ_2. Taking into account the parity properties of the θ-functions, it is immediate to show that $\mathrm{Tr}^{\,\mathbf{(b)}} = \mathrm{Tr}^{\,\mathbf{(a)}}$ in all sectors.

Using (3.52) and the standard results for the four transverse directions, in the case of instantons, i.e. for D3/D(-1) systems, we find in the end that the integrand of (3.50) is proportional to

$$\begin{aligned} &\theta_2(\epsilon_1\tau|\tau)\theta_2(\epsilon_2\tau|\tau)\left[\theta_3(0|\tau)\right]^2 \pm \theta_1(\epsilon_1\tau|\tau)\theta_1(\epsilon_2\tau|\tau)\left[\theta_4(0|\tau)\right]^2 \\ &- \theta_3(\epsilon_1\tau|\tau)\theta_3(\epsilon_2\tau|\tau)\left[\theta_2(0|\tau)\right]^2\,. \end{aligned} \tag{3.53}$$

In the case of D(-1) branes, i.e. with the upper sign in (3.53), this expression identically can vanish only for a self-dual background ($\epsilon_1 = \epsilon_2 = \epsilon$) thanks to the identity

$$\left[\theta_2(\epsilon\tau|\tau)\right]^2\left[\theta_3(0|\tau)\right]^2 + \left[\theta_1(\epsilon\tau|\tau)\right]^2\left[\theta_4(0|\tau)\right]^2 - \left[\theta_3(\epsilon\tau|\tau)\right]^2\left[\theta_2(0|\tau)\right]^2 = 0\,. \tag{3.54}$$

If we consider instead anti-D(-1) branes, i.e. if we take the lower sign in (3.53), we see that the interaction energy vanishes only for an anti-self-dual background $\epsilon_1 = -\epsilon_2 = \epsilon$, since θ_1 is odd in its first argument while θ_2 and θ_3 are even. In conclusion we see that there must be a precise relation between the charge of the D-instantons and the (anti-)self-duality of the B background in order to have a stable brane system.

4. Non-commutative gauge instantons from open strings

In this section we are going to present an explicit realization of instantons in non-commutative gauge theories using the open strings of the D3/D(−1) systems described above, thus extending the analysis of ref. [21] to branes in a background B field. For definiteness, we discuss in detail the stable case of a D3/D(−1) system in a self-dual background, starting from the ADHM constraints.

4.1 The ADHM constraints

The ADHM measure on the instanton moduli space can be derived from scattering amplitudes involving all excitations of the open strings with at least one end point on the D-instantons. As mentioned in section 3.2, the B background does not have any effect on the (−1)/(−1) strings and thus their contribution to the ADHM measure is identical to that of the undeformed (commutative) theory. On the other hand, the mixed (−1)/3 or 3/(−1) sectors feel the presence of the B field, and thus to find the non-commutative modifications in the ADHM measure we have to compute only the amplitudes that involve mixed moduli. For example, let us consider the coupling among $w_{\dot\alpha}$, $\overline{w}_{\dot\alpha}$ and the auxiliary fields D_c^-, which play the rôle of a Lagrange multipliers for the bosonic ADHM constraints. The vertex operators for $w_{\dot\alpha}$ and $\overline{w}_{\dot\alpha}$ are given in (3.45) and its conjugate, while vertex for D_c^- is the same as the undeformed one given in (2.12). However, in the self-dual background it is more convenient to rewrite the latter in the following manner

$$V_D = -2\, D_c^- \, J_c^{(-)} \tag{4.1}$$

using the $\mathrm{SU}(2)_-$ currents defined in (3.39). The coupling among the moduli we are considering is explicitly given by

$$\langle\!\langle\, V_w V_D V_{\overline{w}} \,\rangle\!\rangle \equiv C_0 \int \frac{dy_1 dy_2 dy_3}{dV_{\mathrm{CKG}}}\; \mathrm{tr}\, \langle V_w(y_1) V_D(y_2) V_{\overline{w}}(y_3)\rangle\,, \tag{4.2}$$

where $C_0 = 2/g_0^2$ is the normalization of the mixed disk amplitudes in our present conventions (see *e.g.* ref. [21] for further details). Moreover, dV_{CKG} is the $\mathrm{SL}(2,\mathbb{R})$ invariant volume element, and the trace is over the $\mathrm{U}(N)$ Chan-Paton factors. Using the expression for the various vertex operators, after a few straightforward steps, the amplitude (4.2) becomes

$$\begin{aligned}\langle\!\langle\, V_w V_D V_{\overline{w}} \,\rangle\!\rangle = &- D_c^-\, \mathrm{tr}\left(w_{\dot\alpha}\, \overline{w}_{\dot\beta}\right) (y_1-y_2)(y_2-y_3)(y_3-y_1)\\ &\times \Big[\langle \mathcal{V}^{\dot\alpha}(y_1)\, J_c^{(-)}(y_2)\, \overline{\mathcal{V}}^{\dot\beta}(y_3)\rangle \;+\; \langle \mathcal{V}'^{\dot\alpha}(y_1)\, J_c^{(-)}(y_2)\, \overline{\mathcal{V}'}^{\dot\beta}(y_3)\rangle\Big]\,.\end{aligned} \tag{4.3}$$

As discussed in section 3.3.2, the vertices $\mathcal{V}$ and $\mathcal{V}'$ transform in the same way under the currents $J_c^{(-)}$, and indeed from (3.41) and (3.44a) one can show that

$$\begin{aligned}\langle \mathcal{V}^{\dot\alpha}(y_1)\, J_c^{(-)}(y_2)\, \overline{\mathcal{V}}^{\dot\beta}(y_3)\rangle &= \langle \mathcal{V}'^{\dot\alpha}(y_1)\, J_c^{(-)}(y_2)\, \overline{\mathcal{V}'}^{\dot\beta}(y_3)\rangle\\ &= \frac{\mathrm{i}}{2}\, \frac{(\tau^c)^{\dot\alpha\dot\beta}}{(y_1-y_2)\,(y_1-y_3)\,(y_2-y_3)}\,.\end{aligned} \tag{4.4}$$

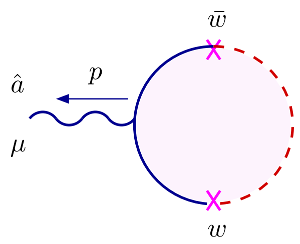


Figure 2: The mixed disk diagram that describes the emission of a gauge vector field $A^{\hat{a}}_{\mu}$ with momentum p represented by the outgoing wavy line.

Inserting this result into (4.3), we simply obtain

$$\langle\!\langle\, V_{\bar{w}} V_D V_w \,\rangle\!\rangle = \mathrm{i}\, D^-_c \,\mathrm{tr}\, \left(w_{\dot{\alpha}} (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \overline{w}^{\dot{\beta}} \right) = \mathrm{i} D^-_c W^c \,, \tag{4.5}$$

which is exactly the same amplitude of the undeformed theory [21].

With similar explicit calculations one can check that all other disk diagrams involving mixed moduli are not affected by the self-dual background, and thus the complete non-commutative ADHM moduli action is the same as in the commuting theory. In particular, the bosonic and fermionic ADHM constraints are still given by (2.13) and (2.14), respectively. As we shall see in the next section, this string result is in agreement with the explicit ADHM construction for non-commutative instantons [30].

4.2 The instanton profile in a self-dual background

In string theory the classical instanton solutions are obtained from 1-point diagrams that describe the emission of the gauge fields from mixed disks [21]. The simplest example of such a mixed disk diagram is represented in figure 2, which describes the emission of a gauge vector.

For simplicity, in the following we discuss only instantons with charge $k = 1$ in the non-commutative U(2) gauge theory, but our analysis can be extended to the general case without any problems. The amplitude described in figure 2 explicitly reads

$$A^{\hat{a}}_{\mu}(p) = \langle\!\langle\, V_{\overline{w}}\, \mathcal{V}^{(0)}_{A^{\hat{a}}_{\mu}}(-p)\, V_w \,\rangle\!\rangle \tag{4.6}$$

where $\mathcal{V}^{(0)}_{A^{\hat{a}}_{\mu}}(-p)$ is the gluon vertex operator in the 0-superghost picture with outgoing momentum and without polarization, i.e.

$$\mathcal{V}^{(0)}_{A^{\hat{a}}_{\mu}}(-p) = 2\mathrm{i}\, T^{\hat{a}} \left(\partial X_\mu \,-\, \mathrm{i}\, p \cdot \psi\, \psi_\mu\right) \mathrm{e}^{-\mathrm{i}p\cdot X} \,, \tag{4.7}$$

where $T^{\hat{a}}$ is the adjoint U(2) Chan-Paton factor. With the insertion of such a vertex, the disk amplitude (4.6) carries the Lorentz structure and the quantum numbers that are appropriate for an emitted gauge vector field. The correlation function in (4.6) receives contribution only from the $\psi^\mu \psi^\nu$ part of (4.7), which again can be conveniently rewritten in terms of the $\mathrm{SU}(2)_\pm$ fermionic currents (3.39). Thus, the relevant part of the gluon vertex is

$$\mathcal{V}^{(0)}_{A^{\hat{a}}_{\mu}}(-p) \sim 2p^\nu\, T^{\hat{a}} \left(\eta^c_{\nu\mu} J^{(+)}_c \,+\, \bar{\eta}^c_{\nu\mu} J^{(-)}_c \right) \mathrm{e}^{-\mathrm{i}p\cdot X} \,. \tag{4.8}$$

JHEP05(2006)069

Notice, that differently from the auxiliary vertex (4.1), the gluon vertex (4.8) depends both on the $\mathrm{SU}(2)_+$ and on the $\mathrm{SU}(2)_-$ currents. This fact has important consequences, as we shall see momentarily.

The calculation of the $J_c^{(-)}$ contribution to the amplitude (4.6) essentially coincides with the one outlined in the previous subsection for the amplitude (4.3) (see also ref. [21] for further details). Since the vertices $\mathcal{V}$ and $\mathcal{V}'$ appearing in V_w and $V_{\overline{w}}$ behave in the same way under $\mathrm{SU}(2)_-$, they produce an identical contribution to the gluon emission which is given by

$$\mathcal{A}_\mu^{\hat{a}\,(-)}(p) = \frac{\mathrm{i}}{2}\,(T^{\hat{a}})^v{}_u\, p^\nu\, \bar{\eta}^c_{\nu\mu} \left(w_{\dot{\alpha}}{}^u\, (\tau_c)^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{w}^{\dot{\beta}}{}_v \right) . \tag{4.9}$$

Imposing the bosonic ADHM constraints (2.13) on w and $\overline{w}$, we can show that the matrices

$$(T_c)^u{}_v \equiv \frac{1}{2\rho^2} \left(w_{\dot{\alpha}}{}^u\, (\tau_c)^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{w}^{\dot{\beta}}{}_v \right) , \tag{4.10}$$

where $\rho^2 \equiv (\bar{w}^{\dot{\alpha}}{}_u\, w_{\dot{\alpha}}{}^u)/2$, satisfy the SU(2) algebra, so that (4.9) can be rewritten as

$$\mathcal{A}_\mu^{\hat{a}\,(-)}(p) = \mathrm{i}\rho^2\, \mathrm{Tr}\, (T^{\hat{a}}\, T_c)\, p^\nu\, \bar{\eta}^c_{\nu\mu}\,. \tag{4.11}$$

Decomposing the adjoint U(2) index $\hat{a} = (0, a)$ into its U(1) and SU(2) parts, we see that only the SU(2) components are non-vanishing, namely

$$\mathcal{A}_\mu^{a\,(-)}(p) = -\frac{\mathrm{i}}{2}\,\rho^2\, \bar{\eta}^a_{\mu\nu} p^\nu \quad , \qquad \mathcal{A}_\mu^{0\,(-)}(p) = 0\,. \tag{4.12}$$

Things are different instead for the $J_c^{(+)}$ contributions to the amplitude (4.6): this is precisely the point where the non-trivial structure of the vertices V_w and $V_{\overline{w}}$ given in (3.45) shows its relevance. In fact, while the correlators of $\mathcal{V}$ and $\mathcal{V}'$ with the step operators $J_\pm^{(+)}$ are vanishing, i.e.

$$\langle \overline{\mathcal{V}}_{\dot{\beta}}(y_1)\, J_\pm^{(+)}(y_2) \mathcal{V}^{\dot{\alpha}}(y_3)\rangle \;=\; \langle \overline{\mathcal{V}'}_{\dot{\beta}}(y_1)\, J_\pm^{(+)}(y_2) \mathcal{V}'^{\dot{\alpha}}(y_3)\rangle = 0\,, \tag{4.13}$$

their correlators with $J_3^{(+)}$ are instead non-trivial, as one can see from the OPE's (3.42) and (3.44b). Indeed we have

$$\begin{aligned} \langle \overline{\mathcal{V}}_{\dot{\beta}}(y_1)\, J_3^{(+)}(y_2) \mathcal{V}^{\dot{\alpha}}(y_3)\rangle &= -\,\langle \overline{\mathcal{V}'}_{\dot{\alpha}}(y_1)\, J_3^{(+)}(y_2) \mathcal{V}'^{\dot{\alpha}}(y_3)\rangle \\ &= \frac{\mathrm{i}\,\epsilon\, \delta^{\dot{\alpha}}_{\dot{\beta}}}{(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)}\,. \end{aligned} \tag{4.14}$$

Thus, the $J_c^{(+)}$ part of the gluon emission is

$$\mathcal{A}_\mu^{\hat{a}\,(+)}(p) = \pm\,\frac{\mathrm{i}\,\epsilon}{2}\, p^\nu \eta^3_{\mu\nu} (T^{\hat{a}})^v{}_u \bar{w}^{\dot{\alpha}\,u}\, w_{\dot{\alpha}\,v} \tag{4.15}$$

where the + and − signs apply to the contributions of the $\mathcal{V}$ and $\mathcal{V}'$ vertices, respectively. If we impose the bosonic ADHM constraints, we can show that the matrix $\bar{w}^{\dot{\alpha}\,u}\, w_{\dot{\alpha}\,v}$ is proportional to the identity, i.e. $\bar{w}^{\dot{\alpha}\,u}\, w_{\dot{\alpha}\,v} \;=\; \rho^2\, \delta^u_v$, and so only the U(1) part of (4.15) is non-vanishing, namely

$$\mathcal{A}^{0\,(+)}_{\mu}(p) = \pm\,\mathrm{i}\,\epsilon\, p^{\nu}\eta^{3}_{\mu\nu}\rho^{2} \;\; , \qquad \mathcal{A}^{a\,(+)}_{\mu}(p) = 0\,. \tag{4.16}$$

While the result (4.12) is somehow expected (and in agreement with field theory calculations), the presence of a U(1) component of the form (4.16) is puzzling since one does not expect a non-vanishing abelian part in the gluon emission at this order. However, in our string realization of the non-commutative ADHM instantons the vertices $\mathcal{V}$ and $\mathcal{V}'$, which correspond to the two types of Neumann boundary conditions in the presence of a B field, are treated in a symmetric way and their respective contributions to a given amplitude must be added together. Thus, the complete $J^{(-)}$ piece of the gluon emission is simply twice the result given in (4.12) for each sector, while the total $J^{(+)}$ contribution vanishes because the $\mathcal{V}$ and $\mathcal{V}'$ parts exactly cancel each other. In other words the full amplitude (4.6) is

$$A^{a}_{\mu}(p) = -\,\mathrm{i}\,\rho^{2}\,\bar{\eta}^{a}_{\mu\nu}p^{\nu} \;\; , \qquad A^{0}_{\mu}(p) = 0\,. \tag{4.17}$$

As explained in ref. [21], to obtain the space-time profile of the instanton we must take the Fourier transform of the momentum space amplitude after inserting a gluon propagator $\delta_{\mu\nu}/p^2$; in this way we get

$$A^{a}_{\mu}(x) = \int \frac{d^4p}{(2\pi)^2}\, A^{a}_{\mu}(p)\,\frac{1}{p^2}\,\mathrm{e}^{\mathrm{i}p\cdot x} = 2\rho^{2}\,\bar{\eta}^{a}_{\mu\nu}\,\frac{x^{\nu}}{|x|^{4}} \;\; , \qquad A^{0}_{\mu}(x) = 0\,. \tag{4.18}$$

In the following section we will see that (4.18) represents the leading term in the large distance expansion ($|x| \gg \rho$) of the classical solution in the singular gauge for an instanton of size ρ and charge $k = 1$ in the non-commutative U(2) theory. The fact that the instanton field is in the singular gauge is not surprising since in our D-brane set-up gauge instantons arise from D-instantons which are point-like objects localized inside the world-volume of the D3 branes [21].

Notice that the gauge field in (4.18) does not depend on the non-commutativity parameter θ and is the same as the leading term at large distance of the BPST instanton of the ordinary SU(2) Yang-Mills theory in the singular gauge [37]. However, the presence of a non-trivial B background is not irrelevant and it shows up in the sub-leading terms of the instanton solution. Indeed, higher order contributions in the large distance expansion of the instanton profile can be obtained by sewing the leading source term with the vertices of the non-commutative gauge theory, as indicated for example in figure 3. There, two gauge vector fields emitted from two disks recombine through the non-commutative 3-gluon vertex and yield the second order correction to the instanton profile. Since the non-commutative vertex contains a part proportional to the $d_{\hat{a}\hat{b}\hat{c}}$-symbols of the gauge group, a gluon can be emitted at second order also along the U(1) direction, even if at the first order only the non-abelian components were produced.

Let us consider in particular the diagram *a)* in which the two disks are sewn together with the part of the 3-gluon vertex that is proportional to the structure constants of the gauge group (see figure 1). This diagram yields a second order correction to the SU(2) field

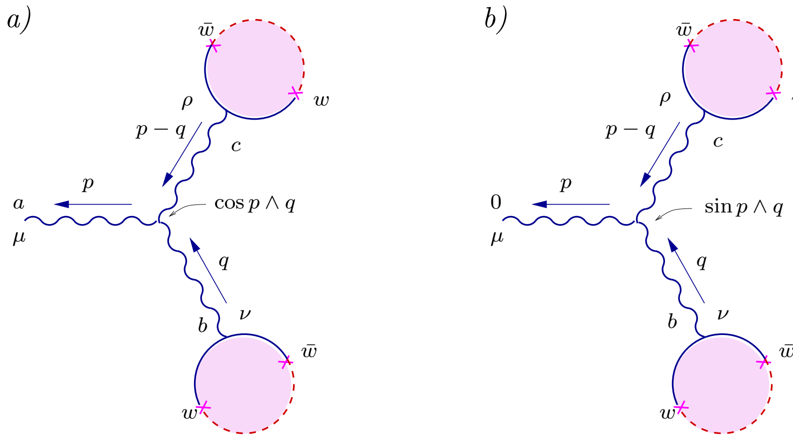


Figure 3: The diagrams that account for sub-leading corrections in the large-distance expansion of the gluon emission amplitude. The diagram *a)* refers to the emission of the SU(N) gluon, while the diagram *b)* the U(1) part. The 3-gluon vertices to be used, here only symbolically indicated, are more explicitly described in figure 1.

given by

$$
\begin{aligned}
A^a_\mu(p)^{(2)} = \frac{\mathrm{i}}{2}\int \frac{d^4q}{(2\pi)^2}\epsilon^{abc}\cos{(p\wedge q)}\Big[(p-2q)_\mu\delta_{\nu\rho} + (p+q)_\rho\delta_{\mu\nu} + (q-2p)_\nu\delta_{\rho\mu}\Big] \\
\times\frac{1}{q^2}\,A^b_\nu(q)^{(1)}\,\frac{1}{(p-q)^2}\,A^c_\rho(p-q)^{(1)}\,,
\end{aligned}
\tag{4.19}
$$

where we have included a symmetry factor of $1/2$ and denoted with a superscript $^{(1)}$ the first-order fields of (4.18), and by $^{(2)}$ the second-order correction we are computing. Using (4.18), after some standard algebra, we find

$$
A^a_\mu(p)^{(2)} = -\mathrm{i}\rho^4\overline{\eta}^a_{\mu\nu}\int\frac{d^4q}{(2\pi)^2}\frac{\cos{(p\wedge q)}}{q^2(p-q)^2}\,\Big[(2p\cdot q-p^2)\,q^\nu + (p\cdot q-2q^2)\,p^\nu\Big]\,. \tag{4.20}
$$

The integral over q can be computed in dimensional regularization upon expanding

$$
\cos{(p\wedge q)} = 1-\frac{1}{2}\Big(\frac{1}{2}p_\mu\theta^{\mu\nu}q_\nu\Big)^2+\dots\ , \tag{4.21}
$$

where θ is the non-commutativity parameter (3.10), and the result in d dimensions is

$$
A^a_\mu(p)^{(2)} = \mathrm{i}\rho^4\overline{\eta}^a_{\mu\nu}p^\nu(p^2)^{\frac{d}{2}-1}\sum_{m=0}^{\infty}\frac{(-|p|^4)^m}{m!}\frac{\pi\,\Gamma\left(m+\frac{d}{2}\right)}{2^{\frac{d}{2}}\sin\left(\pi(m+\frac{d}{2})\right)\Gamma(2m+d-1)}\left(\frac{\theta^2}{16}\right)^m .
$$

Taking the Fourier transform after inserting the gluon propagator, we find

$$
\begin{aligned}
A^a_\mu(x)^{(2)} &= \lim_{d\to4}\int\frac{d^dp}{(2\pi)^{\frac{d}{2}}}\,A^c_\mu(p)^{(2)}\,\frac{1}{p^2}\,\mathrm{e}^{\mathrm{i}p\cdot x} \\
&= -2\rho^4\overline{\eta}^a_{\mu\nu}\frac{x^\nu}{|x|^6}\sum_{m=0}^{\infty}(m+1)(2m)!\left(\frac{\theta}{|x|^2}\right)^{2m} \\
&= -2\rho^4\overline{\eta}^a_{\mu\nu}\frac{x^\nu}{|x|^6}\left(1+\frac{4\theta^2}{|x|^4}+\dots\right)\,.
\end{aligned}
\tag{4.22}
$$

In the first term of the last line we recognize the sub-leading term in the large distance expansion ($|x| \gg \rho$) of the standard SU(2) instanton in the singular gauge (see for example eq. (4.16) of ref. [21]), while the second term represents the non-commutative deformation. In the next section we will show that this result is in agreement with the explicit non-commutative ADHM instanton solution. Notice, however, that the series (4.22) is divergent and thus it must be interpreted as an asymptotic expansion for $\theta/|x|^2 \to 0$. As a consequence, possible non-perturbative terms, like $\mathrm{e}^{-|x|^2/\theta}$ for instance, cannot be accounted in this approach.

Let us now consider the diagram *b)* of figure 3, which is responsible for a U(1) component in the instanton profile that is absent in the commutative case. Using the explicit expression for the 3-gluon vertex, we find

$$\begin{aligned} A^0_\mu(p)^{(2)} &= \frac{\mathrm{i}}{2}\int\frac{d^4q}{(2\pi)^2}\,\delta^{bc}\,\sin(p\wedge q)\,\Big[(p-2q)_\mu\,\delta^\rho_\nu + (p+q)^\rho\,\delta_{\mu\nu} + (q-2p)_\nu\,\delta^\rho_\mu\Big] \\ &\quad\times\frac{1}{q^2}\,A^b_\nu(q)^{(1)}\,\frac{1}{(p-q)^2}\,A^c_\rho(p-q)^{(1)}\,, \end{aligned} \tag{4.23}$$

which after some standard algebra becomes

$$A^0_\mu(p)^{(2)} = \mathrm{i}\rho^4\int\frac{d^4q}{(2\pi)^2}\,\frac{\sin(p\wedge q)}{q^2(p-q)^2}\,\Big[p_\mu(q\cdot p - 3q^2) + q_\mu(6q^2 - 6q\cdot p + 2p^2)\Big]\,. \tag{4.24}$$

Again, the integral over q can be computed in dimensional regularization after expanding $\sin(p\wedge q)$ in powers of θ. Finally, taking the Fourier transform of the resulting terms, we obtain the following space-time dependence for the U(1) field

$$\begin{aligned} A^0_\mu(x)^{(2)} &= -\rho^4\frac{\theta_{\mu\nu}x^\nu}{|x|^8}\sum_{m=0}^{\infty}(m+1)(2m+1)!\left(\frac{\theta}{|x|^2}\right)^{2m} \\ &= -\rho^4\frac{\theta_{\mu\nu}x^\nu}{|x|^8} + \ldots\,. \end{aligned} \tag{4.25}$$

In the next section we will show that this expression is in perfect agreement with the abelian part of the non-commutative U(2) ADHM instanton solution in the singular gauge (again up to non-perturbative terms in $\theta/|x|^2$).

Our analysis can be generalized also to mixed diagrams that contain insertions of fermionic mixed moduli and are responsible for the emission of the other components of the gauge vector multiplet, along the lines discussed in ref. [21]. In this way one can reconstruct the full non-commutative superinstanton solution from string theory.

5. The ADHM construction for non-commutative instantons in the singular gauge

The classical profile for non-commutative instantons can be derived from a generalization of the standard ADHM construction in which the non-commutative nature of space-time is properly taken into account [30]. In this section we are going to briefly present such a construction and, in order to match with the string results we have obtained so far, we will consider specifically the non-commutative instantons in the singular gauge.

JHEP05(2006)069

In the standard ADHM construction of a self-dual instanton with charge $k = 1$ in the U(N) gauge theory (see, for instance, ref. [20] for a review), the basic object is a $(N+2)\times 2$ complex valued matrix $\Delta(x)$ defined as

$$\Delta(x) = a + b\,x \tag{5.1}$$

where a and b can always be put in the form

$$a = \begin{pmatrix} w \\ -a' \end{pmatrix} \qquad\qquad b = \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} \tag{5.2}$$

in which the upper components are $N \times 2$ matrices and the lower components are 2×2 matrices. Finally, in (5.1) x stands for the 2×2 matrix $x_\mu\,\sigma^\mu$ defined in terms of the SO(4) spinor matrices. The variables w and a' appearing in (5.2) are precisely some of the ADHM moduli for which we have presented an explicit string realization in sections 2 and 3. Let $\overline{\Delta}(x) = \Delta^\dagger(x)$ and denote by $U(x)$ a null vector of $\overline{\Delta}(x)$, i.e. a solution of $\overline{\Delta}(x)\,U(x) = 0$. Then, if the following completeness and factorization constraints

$$\Delta(x)\,f(x)\,\overline{\Delta}(x) \;+\; U(x)\,\overline{U}(x) = \mathbb{1} \tag{5.3a}$$

$$\overline{\Delta}(x)\,\Delta(x) = f^{-1}(x)\mathbb{1} \tag{5.3b}$$

are satisfied for some (arbitrary) function $f(x)$, one can show that the gauge potential

$$A_\mu = \mathrm{i}\,\overline{U}(x)\,\partial_\mu\,U(x) \tag{5.4}$$

has a self-dual field strength and thus describes an instanton.

This ADHM construction can be generalized to non-commutative gauge theories [30]. The only formal change in the above equations is that now x^μ has to be interpreted as an operator $\hat{x}^\mu$, subject to the following commutation rules

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\ , \tag{5.5}$$

where θ is the non-commutativity parameter. In particular this implies that (5.4) must be replaced by

$$A_\mu = \mathrm{i}\,\overline{U}(\hat{x})\,[\hat{\partial}_\mu\,,\,U(\hat{x})]\,. \tag{5.6}$$

Let us consider, as a specific example, the non-commutative theory with gauge group U(2), so that we can compare this formal construction with the explicit results obtained from string theory in section 4. In this case the constraint (5.3b) imposes a reality condition on a' and the following relation on w and $\overline{w}$

$$w_{\dot\alpha}{}^{u}\,(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\overline{w}^{\dot\beta}{}_{u} - \overline{\eta}^{c}_{\mu\nu}\theta^{\mu\nu} = 0\,. \tag{5.7}$$

This is the non-commutative version of the bosonic ADHM constraint (2.13) for $k = 1$ and gauge group U(2). If $\theta^{\mu\nu}$ is self-dual, the last term vanishes and (5.7) reduces simply to $W^c = 0$. We therefore confirm explicitly the string results of section 4.1, where we have shown that the ADHM constraints are not affected by a self-dual background.

The constraints $W^c = 0$ can be solved simply by setting $w_{u\dot\alpha} = \rho\delta_{u\dot\alpha}$, so that we have

$$\Delta(\hat x) = \begin{pmatrix} \rho \\ \hat x - a' \end{pmatrix}, \qquad \overline{\Delta}(\hat x) = \begin{pmatrix} \rho \ , \ \bar{\hat x} - \bar{a}' \end{pmatrix} . \tag{5.8}$$

With this choice one can check that eq. (5.3b) is satisfied with

$$\frac{1}{f(\hat x)} = \rho^2 + |\hat x - a'|^2 \tag{5.9}$$

where $|\hat x - a'|^2 = \sum_\mu (\hat x_\mu - a'_\mu)^2$.

To proceed further, we need to find a null vector for $\overline{\Delta}(\hat x)$. A solution of $\overline{\Delta}(\hat x)\, U(\hat x) = 0$ has been proposed in ref. [30] where a non-commutative instanton configuration has been obtained in the regular gauge. However, to make contact with the string theory realization of the gauge instantons presented in sections 3 and 4, we need to have the gauge vector field in the singular gauge since the entire instanton charge is concentrated at the locations of the D(-1) branes. Therefore, we now present a different expression for $U(\hat x)$ that we derive by mimicking what is usually done in commutative ADHM construction to obtain the instanton profile in the singular gauge. More specifically, we consider the matrix

$$U(\hat x) = \begin{pmatrix} \frac{|\hat x - a'|}{\sqrt{\rho^2 + |\hat x - a'|^2}} \\ -\rho(\hat x - a')\, \frac{1}{|\hat x - a'|\sqrt{\rho^2 + |\hat x - a'|^2}} \end{pmatrix} \tag{5.10}$$

which is the straightforward generalization of the matrix used in the commutative case, where the coordinates x^μ have been replaced by the operators $\hat x^\mu$. With some simple manipulations one can check that $U(\hat x)$ is a null vector of $\overline{\Delta}(\hat x)$. However, it does not satisfy the completeness constraint (5.3a), as noticed[10] in [31], in agreement with the impossibility of finding a gauge transformation from the regular to the singular gauge in non-commutative theories [30]. Despite this fact, the matrix (5.10) can still be used to obtain an instanton profile under special conditions.

To see this, it is useful to exploit the one-to-one correspondence between operators depending on the non-commuting $\hat x^\mu$'s and ordinary functions of x^μ multiplied with the Moyal product

$$F(x) \star G(x) \equiv F(x) \exp\left\{ \frac{\mathrm{i}}{2}\, \theta^{\mu\nu}\, \overleftarrow{\partial_\mu}\, \overrightarrow{\partial_\nu} \right\} G(x)\,. \tag{5.11}$$

The precise correspondence is obtained by taking the Fourier transform of a function of x^μ and anti-transforming it back with $\mathrm{e}^{-\mathrm{i}k\cdot\hat x}$. In particular, following this rule one can show that x^μ corresponds simply to $\hat x^\mu$, and more generally that to any function it corresponds an operator constructed with the complete symmetrization of the $\hat x^\mu$'s. For example, the function $x^\mu x^\nu$ corresponds to $\frac{1}{2}\,[\hat x^\mu \hat x^\nu + \hat x^\nu \hat x^\mu]$, while $|x - a'|^2$ corresponds to $|\hat x - a'|^2$. Along these lines, it is possible to show that

$$\frac{1}{(\rho^2 + |\hat x - a'|^2)^\alpha} \quad \leftrightarrow \quad 2\,\frac{(2|\theta|)^{-\alpha}}{\Gamma(\alpha)} \int_0^1 dt \left(\frac{1-t}{1+t}\right)^{\frac{\rho^2}{2|\theta|}} \frac{\mathrm{e}^{-t\frac{|x-a'|^2}{|\theta|}}}{\log^{1-\alpha}\left(\frac{1+t}{1-t}\right)} \tag{5.12}$$

[10]See however [32], where a modification of the ansatz is proposed.

for any $\alpha > 0$, from which (for $\rho = a' = 0$ and $\alpha = 1$) it follows that

$$\frac{1}{|\hat{x}|^2} \leftrightarrow \frac{1}{|x|^2}\left(1 - \mathrm{e}^{-\frac{|x|^2}{|\theta|}}\right) . \tag{5.13}$$

Let us now apply this correspondence to the matrix $U(\hat{x})$ in (5.10) and fix, for simplicity, the instanton center in the origin by setting $a' = 0$. Then, one can show that

$$\Delta(x) \star f(x) \star \overline{\Delta}(x) + U(x) \star \overline{U}(x) = \mathbb{1} + D(x)\,, \tag{5.14}$$

i.e. the non-commutative completeness relation (5.3a) is violated by

$$D(x) = \begin{pmatrix} 0 & 0 \\ 0 & d(x) \end{pmatrix} \qquad \text{where} \qquad d(x) = -2\,\mathrm{e}^{-\frac{|x|^2}{|\theta|}} \begin{pmatrix} 1 + \frac{\theta}{|\theta|} & 0 \\ 0 & 1 - \frac{\theta}{|\theta|} \end{pmatrix} . \tag{5.15}$$

Notice, however, that this violating term vanishes asymptotically in the large distance expansion $|x|^2/|\theta| \to \infty$ and is non-perturbative in the non-commutativity parameter. Thus, if we want to establish a connection with the string results of section 4 that have been obtained in the large distance approximation and with a perturbative expansion in θ, it is natural to neglect $D(x)$ and use the Moyal counterpart of (5.10) to obtain a non-commutative instanton profile in the singular gauge. Proceeding in this way, after some tedious algebra, we find

$$\begin{aligned} A_\mu(x) &= \left\{ \frac{2\rho^2\,\overline{\eta}^a_{\mu\nu} x^\nu}{|x|^2(\rho^2 + |x|^2)} \left[1 - \frac{\rho^2(8|x|^4 + 5|x|^2\rho^4 + \rho^4)}{2|x|^4(\rho^2 + |x|^2)^3}\,\theta^2 \right] \right\} \frac{\tau^a}{2} \\ &\quad + \left\{ -\frac{\rho^4\,\theta_{\mu\nu} x^\nu}{|x|^4(\rho^2 + |x|^2)^2} \right\} \frac{\mathbb{1}}{2} + \mathcal{O}\left(\theta^3\right) \\ &= \left\{ \frac{2\rho^2}{|x|^4}\,\overline{\eta}^c_{\mu\nu} x^\nu \left[1 - \frac{\rho^2}{|x|^2}\left(1 + \frac{4\theta^2}{|x|^4} \right) \right] \right\} \frac{\tau^a}{2} \\ &\quad + \left\{ -\frac{\rho^4}{|x|^8}\,\theta_{\mu\nu} x^\nu \right\} \frac{\mathbb{1}}{2} + \mathcal{O}\left(\theta^3\right) + \mathcal{O}\left(\rho^6\right) \end{aligned} \tag{5.16}$$

where in the second step we have supposed $\frac{\rho^2}{|x|^2} \sim \frac{|\theta|}{|x|^2} \ll 1$. In our normalization, the quantities in braces represent the SU(2) and U(1) components A^a_μ and A^0_μ of the gauge connection, which completely agree with the string theory results presented in section 4.2, and in particular in eqs. (4.18), (4.22) and (4.25). Thus, this analysis confirms that the gluon emission amplitude (4.6) from a mixed disk is the correct source for the non-commutative gauge instantons.

6. Conclusions

As we have explained in the previous sections, our string realization of the non-commutative (anti-)instantons requires a (anti-)self-dual background B field. In fact, only in such a background the D3/D(-1) is stable and allows for an exact conformal field theory description in terms of twisted fields. Therefore, it is natural to ask what happens in a generic background and to what extent the D brane realization can be used in this case. To answer this question one can use a perturbative approach, similarly to what has been done for the RR backgrounds giving rise to non-anti-commutative deformations [34]. In other words one can treat a generic B background as a perturbation around flat space and deduce its effects on the gauge instantons by computing mixed amplitudes with insertions of open string vertex operators, describing the ADHM moduli, and of the closed string vertex operator, describing a constant NS-NS B field. Up to a suitable normalization, the latter is (in the (-1) superghost picture)

$$V_B \;\simeq\; B_{\mu\nu}\left(\psi_L^{\mu}\,\mathrm{e}^{-\phi_L}\right)\left(\psi_R^{\nu}\,\mathrm{e}^{-\phi_R}\right) \tag{6.1}$$

where the subscripts L and R denote the left and right moving parts of the closed string coordinates and superghosts.

The simplest mixed open/closed string diagram involves one vertex V_D for the auxiliary moduli D_C^- given in (2.12) and one closed string vertex V_B. Using standard conformal field theory methods, one finds that the amplitude under consideration is

$$\begin{aligned}\langle\!\langle\, V_D V_B \,\rangle\!\rangle &\equiv C_0 \int \frac{dy\, d^2z}{dV_{\mathrm{CKG}}}\, \langle V_D(y)\, V_B(z,\overline{z})\rangle \\ &\simeq \frac{\mathrm{i}}{g_0^2\,(2\pi\alpha')}\, D_c^-\, \overline{\eta}^c_{\mu\nu} B^{\mu\nu}\end{aligned} \tag{6.2}$$

where we have explicitly exhibited all dimensional constants, but neglected numerical factors which could be absorbed into the normalization of the vertex V_B.

The amplitude (6.2) turns out to be the only one that is relevant in the limit $\alpha' \to 0$ with g_0 kept fixed, which is the appropriate field theory limit for disk diagrams involving open strings with at least one end-point on the D-instantons [21]. Indeed, all other amplitudes with different open string vertex operators or with more insertions of the closed string vertex V_B either vanish or are sub-leading with respect to (6.2) in the field theory limit. Using (2.9) and (3.10), we can rewrite the above result as

$$\langle\!\langle\, V_D V_B \,\rangle\!\rangle \;\simeq\; \mathrm{i}\, D_c^-\, \overline{\eta}^c_{\mu\nu}\,(2\pi\alpha' B^{\mu\nu}) \;=\; \mathrm{i}\, D_c^-\, \overline{\eta}^c_{\mu\nu}\theta^{\mu\nu} \tag{6.3}$$

which shows that the bosonic ADHM constraint (2.13) gets modified by the addition of a term proportional to $\overline{\eta}^c_{\mu\nu}\,\theta^{\mu\nu}$. Of course, if the B background is self-dual this term vanishes in agreement with what we already found in section 4.1. On the contrary, in an anti-self-dual background the deformation corresponding to (6.3) is non-trivial and agrees with the explicit ADHM analysis for non-commutative gauge theories.

In conclusion we have shown that the D3/D(-1) system in a NS-NS B field correctly describes the instantons of non-commutative gauge theories. When the background and the gauge field strengths have the same duality, the brane system is stable and the non-commutative deformation can be treated exactly by means of a twisted conformal field theory; in the other cases only a perturbative string approach is available, but in the field theory limit this always agrees with the non-commutative ADHM construction. Finally, it would be interesting to explore, both in string and in field theory, the meaning of the non-perturbative corrections in $|\theta|/x^2$ to the matrix $U(x)$ of the non-commutative ADHM construction that are needed to satisfy exactly the completeness relation in the singular gauge.

JHEP05(2006)069

Acknowledgments

We thank Francesco Fucito and Igor Pesando for many useful discussions.

A. Notation and conventions

Spinor notation in Euclidean $\mathbb{R}^4$: we consider the Euclidean space $\mathbb{R}^4$ with coordinates x^μ, $\mu = 1, \cdots 4$. The Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ is satisfied by

$$\gamma^\mu = (\gamma^\mu)^\dagger = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.1}$$

where $\sigma^\mu = (\mathrm{i}\vec{\tau}, \mathbb{1})$, $\bar{\sigma}^\mu = (\sigma^\mu)^\dagger = (-\mathrm{i}\vec{\tau}, \mathbb{1})$, with $\vec{\tau}$ being the Pauli matrices.

The matrices $\bar{\sigma}^\mu$ and σ^μ, which satisfy the algebra

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2\delta^{\mu\nu}\, \mathbb{1}\,, \tag{A.2}$$

are the Weyl matrices acting, respectively, on chiral and anti-chiral spinors ψ_α and $\psi^{\dot{\alpha}}$.

In our basis, the charge conjugation matrix C is block diagonal and is given by

$$C = -\begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}\,. \tag{A.3}$$

where $\epsilon^{12} = \epsilon_{12} = -\epsilon^{\dot{1}\dot{2}} = -\epsilon_{\dot{1}\dot{2}} = 1$. We use the convention that the dotted indices are contracted in the $\nearrow$ direction, while undotted ones contracted in the $\searrow$ direction:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta\,, \qquad \psi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}}\,. \tag{A.4}$$

Out of the γ matrices we may construct the generators of the so(4) algebra

$$\Sigma^{\mu\nu} = \frac{1}{4}\,[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \frac{1}{2}\sigma^{\mu\nu} & 0 \\ 0 & \frac{1}{2}\bar{\sigma}^{\mu\nu} \end{pmatrix} \tag{A.5}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2}\big(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\big)\,, \qquad \bar{\sigma}^{\mu\nu} = \frac{1}{2}\big(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\big) \tag{A.6}$$

which are respectively self-dual and anti-self-dual. They act on the irreducible spinor representations corresponding, respectively, to chiral and anti-chiral spinor, as follows

$$\delta_\omega\psi_\alpha = \frac{1}{4}\omega_{\mu\nu}\,(\sigma^{\mu\nu})_\alpha{}^\beta\psi_\beta\,, \qquad \delta_\omega\psi^{\dot{\alpha}} = \frac{1}{4}\omega_{\mu\nu}\,(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\psi^{\dot{\beta}}\,. \tag{A.7}$$

Furthermore, using the symmetry properties of the matrices $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$, one can check that also ψ^β and $\psi_{\dot{\beta}}$ transform as chiral and anti-chiral spinors:

$$\delta_\omega\psi^\alpha = -\frac{1}{4}\omega_{\mu\nu}\,\psi^\beta(\sigma^{\mu\nu})_\beta{}^\alpha\,, \qquad \delta_\omega\psi_{\dot{\alpha}} = -\frac{1}{4}\omega_{\mu\nu}\,\psi_{\dot{\beta}}(\overline{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\,. \tag{A.8}$$

The anti-hermiticity of the $\sigma^{\mu\nu}$ matrices: $\left[(\sigma^{\mu\nu})_\alpha{}^\beta\right]^* = -(\sigma^{\mu\nu})_\beta{}^\alpha$ implies instead that $(\psi_\alpha)^*$ transform like ψ^α. Then, consistently with the conjugation property of σ^μ, we define the complex conjugation as

$$\begin{aligned} (\psi^\alpha)^* &= \overline{\psi}_\alpha\,, \qquad & (\psi_{\dot{\alpha}})^* &= -\overline{\psi}^{\dot{\alpha}} \\ (\psi_\alpha)^* &= -\overline{\psi}^\alpha\,, \qquad & (\psi^{\dot{\alpha}})^* &= \overline{\psi}_{\dot{\alpha}}\,. \end{aligned} \tag{A.9}$$

't Hooft symbols: the Spin(4) group is isomorphic to $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. The isomorphism is described at the level of generators using the 't Hooft symbols:

$$J_{\mu\nu} = \eta^c_{\mu\nu} J^+_c + \overline{\eta}^c_{\mu\nu} J^-_c \ , \tag{A.10}$$

where $J_{\mu\nu}$ ($\mu, \nu = 1, \dots, 4$) are the Spin(4) generators and $J^\pm_c$ ($c = 1, 2, 3$) are the $\mathrm{SU}(2)_\pm$ generators. Explicitly, the 't Hooft symbols are defined as follows:

$$\overline{\eta}^c_{\mu\nu} = -\overline{\eta}^c_{\nu\mu} \ , \qquad \eta^c_{\mu\nu} = -\eta^c_{\nu\mu} \ , \tag{A.11a}$$

$$\eta^c_{ab} = \overline{\eta}^c_{ab} = \epsilon_{cab} \ , \qquad a, b, c \in \{1, 2, 3\} \ , \tag{A.11b}$$

$$\overline{\eta}^c_{4a} = \eta^c_{a4} = \delta^c_a \tag{A.11c}$$

so that η is self-dual and $\bar{\eta}$ is anti-self-dual.

Applied to the irreducible spinor representations, (A.10) states that chiral and anti-chiral spinors belong respectively to the fundamental representation of $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$:

$$(\sigma_{\mu\nu})_\alpha{}^\beta = \mathrm{i}\eta^c_{\mu\nu} (\tau^c)_\alpha{}^\beta \qquad , \qquad (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \mathrm{i}\overline{\eta}^c_{\mu\nu} (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \ . \tag{A.12}$$

Let us collect here some useful formulae for manipulating the 't Hooft symbols:

$$\eta^c_{\mu\nu}\eta^d_{\mu\nu} = \overline{\eta}^c_{\mu\nu}\overline{\eta}^d_{\mu\nu} = 4\delta^{cd} \tag{A.13}$$

$$\eta^c_{\mu\nu}\eta^c_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma} + \epsilon_{\mu\nu\rho\sigma} \tag{A.14}$$

$$\overline{\eta}^c_{\mu\nu}\overline{\eta}^c_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma} - \epsilon_{\mu\nu\rho\sigma} \tag{A.15}$$

We also have

$$\epsilon_{abc}\overline{\eta}^b_{\nu\sigma}\overline{\eta}^c_{\rho\tau} = \overline{\eta}^a_{\sigma\tau}\delta_{\nu\rho} + \overline{\eta}^a_{\nu\rho}\delta_{\sigma\tau} - \overline{\eta}^a_{\nu\tau}\delta_{\sigma\rho} - \overline{\eta}^a_{\sigma\rho}\delta_{\nu\tau} \ , \tag{A.16}$$

and similarly for the $\eta^b_{\nu\sigma}$ symbols.

References

[1] A. Connes, M.R. Douglas and A.S. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *JHEP* **02** (1998) 003 [hep-th/9711162].

[2] A. Abouelsaood, C.G. Callan Jr., C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys.* **B 280** (1987) 599.

[3] M.M. Sheikh-Jabbari, *More on mixed boundary conditions and D-branes bound states*, *Phys. Lett.* **B 425** (1998) 48 [hep-th/9712199].

[4] C.S. Chu and P.M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219].

[5] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030 [hep-th/9903205].

[6] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Dirac quantization of open strings and noncommutativity in branes*, *Nucl. Phys.* **B 576** (2000) 578 [hep-th/9906161].

[7] C.S. Chu and P.M. Ho, *Constrained quantization of open string in background B field and noncommutative D-brane*, *Nucl. Phys.* **B 568** (2000) 447 [hep-th/9906192].

JHEP05(2006)069

[8] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[9] C.S. Chu, *Noncommutative open string: neutral and charged*, hep-th/0001144.

[10] P. Haggi-Mani, U. Lindström and M. Zabzine, *Boundary conditions, supersymmetry and A-field coupling for an open string in A B-field background*, *Phys. Lett.* **B 483** (2000) 443 [hep-th/0004061].

[11] M. Mihailescu, I.Y. Park and T.A. Tran, *D-branes as solitons of an $N = 1$, $D = 10$ non-commutative gauge theory*, *Phys. Rev.* **D 64** (2001) 046006 [hep-th/0011079].

[12] C.-S. Chu, R. Russo and S. Sciuto, *Multiloop string amplitudes with B-field and noncommutative QFT*, *Nucl. Phys.* **B 585** (2000) 193 [hep-th/0004183].

[13] C.S. Chu and R. Russo, *Multiloop noncommutative open string theory and their QFT limit*, *Mod. Phys. Lett.* **A 16** (2001) 211 [hep-th/0102164].

[14] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, *Phys. Lett.* **B 574** (2003) 98 [hep-th/0302078].

[15] H. Ooguri and C. Vafa, *The C-deformation of gluino and non-planar diagrams*, *Adv. Theor. Math. Phys.* **7** (2003) 53 [hep-th/0302109]; *Gravity induced C-deformation*, *Adv. Theor. Math. Phys.* **7** (2004) 405 [hep-th/0303063].

[16] N. Seiberg, *Noncommutative superspace, $N = 1/2$ supersymmetry, field theory and string theory*, *JHEP* **06** (2003) 010 [hep-th/0305248].

[17] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135].

[18] M.R. Douglas, *Gauge fields and D-branes*, *J. Geom. Phys.* **28** (1998) 255 [hep-th/9604198]; *Branes within branes*, hep-th/9512077.

[19] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[20] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063].

[21] M. Billò et al., *Classical gauge instantons from open strings*, *JHEP* **02** (2003) 045 [hep-th/0211250].

[22] M.B. Green and M. Gutperle, *D-instanton induced interactions on a D3-brane*, *JHEP* **02** (2000) 014 [hep-th/0002011].

[23] E. Witten, *Small instantons in string theory*, *Nucl. Phys.* **B 460** (1996) 541 [hep-th/9511030].

[24] N. Nekrasov and A.S. Schwarz, *Instantons on noncommutative $\mathbb{R}^4$ and (2,0) superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[25] K. Furuuchi, *Instantons on noncommutative $\mathbb{R}^4$ and projection operators*, *Prog. Theor. Phys.* **103** (2000) 1043 [hep-th/9912047].

[26] K. Furuuchi, *Topological charge of U(1) instantons on noncommutative $\mathbb{R}^4$*, *Prog. Theor. Phys. Suppl.* **144** (2001) 79–91 [hep-th/0010006].

JHEP05(2006)069

[27] N.A. Nekrasov, *Noncommutative instantons revisited*, *Commun. Math. Phys.* **241** (2003) 143 [hep-th/0010017].

[28] K. Furuuchi, *Dp-D(p+4) in noncommutative Yang-Mills*, *JHEP* **03** (2001) 033 [hep-th/0010119].

[29] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[30] C.-S. Chu, V.V. Khoze and G. Travaglini, *Notes on noncommutative instantons*, *Nucl. Phys.* **B 621** (2002) 101 [hep-th/0108007].

[31] O. Lechtenfeld and A.D. Popov, *Noncommutative 't Hooft instantons*, *JHEP* **03** (2002) 040 [hep-th/0109209].

[32] Y. Tian and C.-J. Zhu, *Comments on noncommutative ADHM construction*, *Phys. Rev.* **D 67** (2003) 045016 [hep-th/0210163].

[33] R. Wimmer, *D0-D4 brane tachyon condensation to a BPS state and its excitation spectrum in noncommutative super Yang-Mills theory*, *JHEP* **05** (2005) 022 [hep-th/0502158].

[34] M. Billò, M. Frau, I. Pesando and A. Lerda, $N = 1/2$ *gauge theory and its instanton moduli space from open strings in RR background*, *JHEP* **05** (2004) 023 [hep-th/0402160].

[35] M. Billò, M. Frau, F. Lonegro and A. Lerda, $N = 1/2$ *quiver gauge theories from open strings with RR fluxes*, *JHEP* **05** (2005) 047 [hep-th/0502084].

[36] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678;
S. Hamidi and C. Vafa, *Interactions on orbifolds*, *Nucl. Phys.* **B 279** (1987) 465.

[37] A.A. Belavin, A.M. Polyakov, A.S. Shvarts and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, *Phys. Lett.* **B 59** (1975) 85;
G. 't Hooft, *Computation of the quantum effects due to a four-dimensional pseudoparticle*, *Phys. Rev.* **D 14** (1976) 3432.

[38] A. Armoni, *Comments on perturbative dynamics of non-commutative Yang-Mills theory*, *Nucl. Phys.* **B 593** (2001) 229 [hep-th/0005208].

[39] L. Bonora and M. Salizzoni, *Renormalization of noncommutative U(n) gauge theories*, *Phys. Lett.* **B 504** (2001) 80 [hep-th/0011088].

JHEP05(2006)069